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ON A PARTIAL DIFFERENTIAL OPERATOR ASSOCIATED WITH THE EIGENVALUE PROBLEM OF THE INVARIANT LAPLACIAN

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Dedicated to Professor Younki Chae on his sixtieth birthday

For $\lambda = 4m(n+m)$, a function $f(z) = (1-|z|^2)^m g(z)$ with $g \in X_\lambda$, the eigenspace of the invariant Laplacian $\tilde{\Delta}$ in the unit ball B_n of $\mathbf{C}^n$, satisfies an elliptic differential equation $\Delta_m f = 0$. We make a study of the operator Δ_m as another way to study $\tilde{\Delta} - 4m(n+m)$. For example, if Z_m denotes the class of all solutions f in $C^2(B_n)$ of $\Delta_m f = 0$, we obtain an L^2-growth condition for the projection of a function in Z_m onto $H(p,q)$, the space of all harmonic homogeneous polynomials on $\mathbf{C}^n$ of degree p in z and of degree q in $\bar{z}$, to be 0 unless either $p \leq m$ or $q \leq m$. This corresponds and gives another way to obtain the L^2-growth condition for a function in X_λ to be in the $\mathcal{M}$-subspace Y_4 of X_λ. Y_4 is the space of pluriharmonic functions in the case $\lambda = 0$.

1. Introduction

Let n be any positive integer. Throughout this thesis, $\mathbf{C}^n$ is the n-dimensional complex space with the inner product

$$< z, w > = \sum_{j=1}^{n} z_j \bar{w}_j \qquad (z, w \in \mathbf{C}^n),$$

and the associated norm

$$|z| = < z, z >^{\frac{1}{2}} \qquad (z \in \mathbf{C}^n).$$

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The inner product and the norm make $\mathbf{C}^n$ an n-dimensional Hilbert space whose open unit ball is denoted by B. Thus B consists of all $z \in \mathbf{C}^n$ with $|z| < 1$. The boundary of B is denoted by ∂B, the set of all $z \in \mathbf{C}^n$ with $|z| = 1$. The unique normalized rotation-invariant measure on ∂B is denoted by σ. The invariant Laplacian $\tilde{\Delta}$ is defined by

$$(\tilde{\Delta} f)(z) = 4(1-|z|^2) \sum_{j,k=1}^{n} (\delta_{jk} - z_j \bar{z}_k) \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k}(z), \qquad f \in C^2(B)$$

where δ_{jk} is the Kronecker's symbol. The term "invariant" means that it commutes with the automorphisms of B:

$$\tilde{\Delta}(f \circ \varphi) = (\tilde{\Delta} f) \circ \varphi$$

for every $f \in C^2(B)$ and $\varphi \in Aut(B)$, the group of all automorphisms of B. For a $\lambda \in \mathbf{C}$, we let X_λ denote the space of all $f \in C^2(B)$ that satisfy $\tilde{\Delta} f = \lambda f$. We only consider the case $\lambda = 4m(n+m)$, where m is a nonnegative integer, since $\mathcal{M}$-subspaces(=closed Möbius invariant subspaces) of X_λ are trivial for other λ's. For a nonnegative integer m, we set $f(z) = (1-|z|^2)^m g(z)$ with $g \in X_\lambda (\lambda = 4m(n+m))$. Then f satisfies a differential equation $\Delta_m f = 0$, where

$$\Delta_m = \sum_{j,k=1}^{n} (\delta_{jk} - z_j \bar{z}_k) \frac{\partial^2}{\partial z_j \partial \bar{z}_k} + m \left(\sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j} + \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right) - m^2.$$

We denote by Z_m the space of all solutions of $\Delta_m f = 0$:

$$Z_m = \left\{ f \in C^2(B) : \Delta_m f = 0 \right\}.$$

Then $g \in X_\lambda(\lambda = 4m(n+m))$ if and only if $f \in Z_m$. In other words, the fact that g is a solution of the equation $\tilde{\Delta} g = 4m(n+m) g$ is equivalent to the fact that $f(z) = (1-|z|^2)^m g(z)$ is a solution of $\Delta_m f = 0$. The operator Δ_m is easier to handle than $\tilde{\Delta}$ is but unfortunately it is not invariant under the compositions of automorphisms of B. We make a study of the operator Δ_m as another way to study $\tilde{\Delta} - 4m(n+m)$ and obtain the corresponding versions on Δ_m of the results on $\tilde{\Delta} - 4m(n+m)$ in [3].

2. Harmonic Homogeneous Polynomials

2.1 $\mathcal{H}_s$. We denote by $\mathcal{H}_s$ the space of all harmonic polynomials homogeneous on $\mathbf{C}^n$ of order s:

$$\mathcal{H}_s \equiv \{f : \Delta f = 0 \text{ and } f(tz) = t^s f(z) \text{ for } t > 0\}.$$

Here Δ denotes the ordinary Laplacian. It is defined to be

$$\Delta = 4\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j}.$$

2.2 $H(p,q)$. $H(p,q)$ denotes the space of all harmonic homogeneous polynomials on $\mathbf{C}^n$ of degree p in z and degree q in $\bar{z}$:

$$H(p,q) \equiv \left\{f \in \mathcal{H}_{p+q} : f(z) = \sum\nolimits_{|\alpha|=p,|\beta|=q} C_{\alpha\beta} z^\alpha \bar{z}^\beta\right\},$$

where $C_{\alpha\beta}$'s are complex constants.

2.3 Projection π_{pq} . For each (p,q), the projection $\pi_{pq} : L^2(\partial B) \to H(p,q)$ is given by the integral kernel K_{pq} defined by

$$\pi_{pq} f(\eta) = \int_{\partial B} K_{pq}(\eta,\zeta) f(\zeta) d\sigma(\zeta), \qquad f \in L^2(\partial B).$$

See[5,12.2.5].

2.4. Some basic facts .
The following properties of the space $H(p,q)$ are well known.

(a) $\mathcal{H}_s = \sum_{p+q=s} H(p,q)$ and $H(p,q)$'s are pairwise orthogonal spaces. [5, 12.2.2]

(b) The linear span of $\bigcup_{s=0}^{\infty} \mathcal{H}_s$ is dense in $C(\partial B)$. [5, 12.1.3]

(c) $L^2(\partial B) = \oplus H(p,q), \qquad 0 \le p,q \le \infty$. [5, 12.2.3]

2.5 Hypergeometric functions . The second order linear differential equation

$$t(1-t)y'' + [\gamma - (\alpha+\beta+1)t]y' - \alpha\beta y = 0 \tag{1}$$

is called the Gauss hypergeometric equation, where α, β, γ are constants. The differential equation (1) has a regular singular point at $t = 0$ and has a unique solution $y = y(t)$ with $y(0) = 1$. By [4], the unique solution $y(t)$ is given by the Gauss hypergeometric series

$$F(\alpha,\beta,\gamma;t) = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k} \frac{t^k}{k!},$$

where $(\alpha)_k = \Gamma(\alpha + k)/\Gamma(\alpha)$.

2.6. Radial derivative $\mathcal{D}^\beta f$. If f is real-analytic in B then f has a homogeneous expansion

$$f(z) = \sum_{k=0}^{\infty} P_k(z, \bar{z}),$$

where P_k is a homogeneous polynomial in z and $\bar{z}$ of total degree k. For $\beta > 0$, we define the radial derivative $\mathcal{D}^\beta f$ of f of order β is defined as

$$\mathcal{D}^\beta f(z) = \sum_k (k+1)^\beta P_k(z, \bar{z}).$$

3. Operator Δ_m

3.1. Proposition. *For $f(z) = (1 - |z|^2)^m g(z)$, $\tilde{\Delta} g = 4m(n + m)g$ is equivalent to $\Delta_m f = 0$.*

Proof. Since $g(z) = (1 - |z|^2)^{-m} f(z)$, we have

$$\begin{aligned}
\frac{\partial g}{\partial z_j} &= m(1-|z|^2)^{-m-1}\bar{z}_j f + (1-|z|^2)^m \frac{\partial f}{\partial z_j}, \\
\frac{\partial g}{\partial \bar{z}_j} &= m(1-|z|^2)^{-m-1} z_j f + (1-|z|^2)^m \frac{\partial f}{\partial \bar{z}_j}, \\
\frac{\partial^2 g}{\partial z_j \partial \bar{z}_k} &= m(m+1)(1-|z|^2)^{-m-2}\bar{z}_j z_k f + m(1-|z|^2)^{-m-1}\delta_{jk} f \\
&\quad + m(1-|z|^2)^{-m-1}\bar{z}_j \frac{\partial f}{\partial \bar{z}_k} + m(1-|z|^2)^{-m-1} z_k \frac{\partial f}{\partial z_j} \\
&\quad (1-|z|^2)^m \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k}.
\end{aligned}$$

Therefore, after some algebraic manipulations, $\tilde{\Delta} g = 4m(n+m)$ is equivalent to

$$\left\{ \sum_{j,k=1}^{n} (\delta_{jk} - z_j \bar{z}_k) \frac{\partial^2}{\partial z_j \partial \bar{z}_k} + m \sum_{j=1}^{n} \left(z_j \frac{\partial}{\partial z_j} + \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right) - m^2 \right\} f(z) = 0.$$

That is, $\Delta_m f = 0$.

We define a Poisson type kernel for the operator Δ_m

$$P_m(z,\eta) = \frac{\Gamma(n+m)^2}{\Gamma(n)\Gamma(n+2m)} \frac{(1-|z|^2)^{n+2m}}{|1-<z,\eta>|^{2n+2m}}, z \in B, \eta \in \partial B.$$

For a function $f \in C(\partial B)$, a solution of the Dirichlet problem for Δ_m is shown to be given by

$$P_m[f](z) = \int_{\partial B} P_m(z,\eta) f(\eta) d\sigma(\eta), \qquad z \in B,$$

in the following proposition.

3.2. Proposition. (a) *If $f \in C(\partial B)$ and $F(z) = P_m[f](z)$, then*

$$\Delta_m F = 0 \tag{1}$$

and

$$\lim_{r \to 1} F(r\zeta) = f(\zeta) \tag{2}$$

uniformly for ζ as $r \to 1$.

(b) *In particular, if $f \in H(p,q)$, then*

$$F(z) = \frac{F(p-m, q-m, p+q+n; r^2)}{F(p-m, q-m, p+q+n; 1)} f(z).$$

Proof. (a) We note that

$$P_m(z,\eta) = \frac{\Gamma(n+m)^2}{\Gamma(n)\Gamma(n+2m)} (1-|z|^2)^m P(z,\eta)^{1+\frac{m}{n}},$$

where $P(z,\eta)$ is the invariant Poisson kernel of $\tilde{\Delta}$ on B. Since $P^{1+\frac{m}{n}} \in X_{4m(n+m)}$, $P_m(z,\eta) \in Z_m$ for a fixed $\eta \in \partial B$, by Proposition 3.1. Therefore,

$$\begin{aligned} (\Delta_m F)(z) &= \Delta_m \left(\int_{\partial B} P_m(z,\eta) f(\eta) d\sigma(\eta) \right) \\ &= \int_{\partial B} \Delta_m P_m(z,\eta) f(\eta) d\sigma(\eta) \\ &= 0. \end{aligned}$$

This proves (1). We note that

$$\begin{aligned} \alpha(n,m,r) &\equiv \int_{\partial B} P_m(r\zeta,\eta) d\sigma(\eta) \\ &= \frac{\Gamma(n+m)^2}{\Gamma(n)\Gamma(n+2m)} F(-m,-m,n;r^2) \longrightarrow 1 \end{aligned}$$

as $r \to 1$ by Theorem 2.3 and Corollary 2.4 of [3] and (9.5.3) of [4]. We also note that

$$\begin{aligned} P_m[f](r\zeta) - f(\zeta) &= P_m[f](z) - \alpha(n,m,r)f(\zeta) + f(\zeta)\{\alpha(n,m,r)-1\} \\ &= \int_{\partial B}(f(\eta)-f(\zeta))P_m(r\zeta,\eta)d\sigma(\eta) + f(\zeta)\{\alpha(n,m,r)-1\} \\ &= I + II. \end{aligned}$$

For a given $\epsilon > 0$, let $\delta > 0$ be such that $|f(\eta) - f(\zeta)| < \epsilon$ for $|\eta - \zeta| < \delta$; δ depends only on ϵ and not on ζ by the uniform continuity of f. We estimate(I) as a sum of two estimates:

$$|(I)| \le \Big(\int_{|\eta-\zeta|>\delta} + \int_{|\eta-\zeta|<\delta}\Big)|f(\eta)-f(\zeta)|P_m(r\zeta,\eta)d\sigma(\eta). \qquad (3)$$

The first integral of (3) is

$$\begin{aligned} &\int_{|\eta-\zeta|>\delta} |f(\eta)-f(\zeta)|\, P_m(r\zeta,\eta)d\sigma(\eta) \\ \le\ & \int_{|\eta-\zeta|>\delta} (|f(\eta)| + |f(\zeta)|)\, P_m(r\zeta,\eta)d\sigma(\eta) \\ \le\ & 2\|f\| \int_{|\eta-\zeta|>\delta} P_m(r\zeta,\eta)d\sigma(\eta) \\ <\ & \epsilon \end{aligned} \qquad (4)$$

for r sufficiently close to 1, since $P_m(r\zeta,\eta) \to 0$ as $r \to 1$ uniformly for $|\eta - \zeta| > \delta$.

The second integral of (3) is

$$\begin{aligned} \int_{|\eta-\zeta|<\delta} |f(\eta)-f(\zeta)|\, P_m(r\zeta,\eta)d\sigma(\eta) &\le \epsilon \int_{|\eta-\zeta|<\delta} P_m(r\eta,\zeta)d\sigma(\eta) \\ &\le \alpha(n,m,r)\epsilon \\ &< 2\epsilon \end{aligned} \qquad (5)$$

for r sufficiently close to 1. For the estimate of (II), we have

$$\begin{aligned} |(II)| &\le \|f\| \cdot |\alpha(n,m,r) - 1| \\ &< \epsilon \end{aligned} \qquad (6)$$

for r sufficiently close to 1. Therefore (2) follows from (3),(4),(5) and (6).
(b) Since every function in $H(p,q)$ is a linear combination of unitary transformations of $z_1^p \bar{z}_2^q$, we let $f(z) = z_1^p \bar{z}_2^q$ and seek a solution of the form

$F(z) = g(r^2)z_1^p\bar{z}_2^q$. A straightforward computation shows that

$$\sum_{j=1}^n \frac{\partial^2 F}{\partial z_j \partial \bar{z}_j} = [r^2 g''(r^2) + (n+p+q)g'(r^2)]z_1^p\bar{z}_2^q,$$

$$\sum_{j,k=1}^n z_j\bar{z}_k \frac{\partial^2 F}{\partial z_j \partial \bar{z}_k} = [r^4 g''(r^2) + (p+q+1)r^2 g'(r^2) + pq g(r^2)]z_1^p\bar{z}_2^q,$$

$$\sum_{j=1}^n z_j \frac{\partial F}{\partial z_j} = [g'(r^2)r^2 + pg(r^2)]z_1^p\bar{z}_2^q,$$

and

$$\sum_{j=1}^n \bar{z}_j \frac{\partial F}{\partial \bar{z}_j} = [g'(r^2)r^2 + qg(r^2)]z_1^p\bar{z}_2^q.$$

Hence

$$\{r^2(1-r^2)g''(r^2) + (n+p+q-(p-m+q-m+1)r^2)g'(r^2)$$
$$-(p-m)(q-m)g(r^2)\}z_1^p\bar{z}_2^q = 0.$$

Therefore, if $\Delta_m F = 0$, g must satisfy the following ordinary differential equation

$$\begin{aligned} &t(1-t)g''(t) + (n+p+q-(p-m+q-m+1)t)g'(t) \\ &\quad -(p-m)(q-m)g(t) = 0. \end{aligned} \tag{7}$$

But this is the hypergeometric equation with parameters $\alpha = p-m, \beta = q-m, \gamma = n+p+q$ and the solutions are constant multiples of $F(p-m, q-m, n+p+q; t)$. Thus

$$F(z) = CF(p-m, q-m, n+p+q; r^2)f(z)$$

and since $F(r\zeta) \to f(\zeta)$ as $r \to 1$, $C = F(p-m, q-m, n+p+q; 1)^{-1}$.

3.3. Operator $L_{p,q,m}$. For $f(z) = y(|z|^2)h(z)$ with $y \in C^2([0,1]), h \in H(p,q)$, $\Delta_m f$ is seen in (7) to have the form

$$(\Delta_m f)(z) = (L_{p,q,m})(|z|^2)h(z)$$

where

$$(L_{p,q,m}y)(t) = t(1-t)y'' + [p+q+n-(p-m+q-m+1)t]y' - (p-m)(q-m)y.$$

The differential equation $L_{p,q,m}y = 0$ is the Gauss differential equation with parameters $\alpha = p - m, \beta = q - m$ and $\gamma = n + p + q$. Therefore it has a unique solution $y = R(p,q,m;t)$ with $y(0) = 1$, where $R(p,q,m;t) = F(p-m, q-m, p+q+n; t)$.
For a function f in $H(p,q)$, $P_m[f](z) = C(p,q,m)R(p,q,m;|z|^2)f(z)$, where $C(p,q,m) = F(p-m, q-m, p+q+n; 1)^{-1}$ by Proposition 3.2.

We define $H(p,q,m)$ by

$$H(p,q,m) = \left\{ f : f(z) = R(p,q,m;|z|^2)h(z), h \in H(p,q) \right\}.$$

and define $\tilde{\pi}_{pq} f$ by

$$(\tilde{\pi}_{pq} f)(z) = (\pi_{pq} f_r)(\zeta), \qquad z = r\zeta,$$

where $f_r(\zeta) = f(r\zeta)$ for $0 \le r < 1$ and for $\zeta \in \partial B$. Then $\tilde{\pi}_{pq}$ projects Z_m onto $H(p,q,m)$. See [6].

The expansion of $P_m(z,\zeta)$ is given as follows. We follow the argument in [3] for the proof.

3.4. Proposition . *If m is a nonnegative integer then*

$$P_m(z,\zeta) = \sum_{p,q=0}^{\infty} G_{p,q,m}(r) K_{pq}(\eta,\zeta), \qquad z = r\eta \in B, \zeta \in \partial B, \qquad (1)$$

where $G_{p,q,m}(r) = C(p,q,m)R(p,q,m;r^2)r^{p+q}$. The series on the right of (1) converges absolutely and uniformly for $\eta, \zeta \in \partial B$ and $0 \le r \le \rho$ for each $\rho < 1$.

Proof. Let $z = r\eta \in B$. Then we obtain

$$\begin{aligned} |G_{p,q,m}(r)| &= |C(p,q,m)R(p,q,m;r^2)|r^{p+q} \\ &= C(p,q,m)F(p-m, q-m, n+p+q; r^2)r^{p+q} \\ &\le F(p-m, q-mn+p+q; 1)^{-1}F(p-m, q-m, n+p+q; 1)r^{p+q} \\ &= r^{p+q} \end{aligned}$$

for $p, q \ge m$. On the other hand, $K_{pq}(\eta,\zeta)$ is uniformly bounded by constant times $(p+q+1)^{2n}$ by [2, p406-407]. Hence

$$\begin{aligned} \sum_{p,q \ge m}^{\infty} |G_{p,q,m}(r)K_{pq}(r\eta)| &\le C(n) \sum_{p,q > N}^{\infty} r^{p+q}(p+q+1)^{2n} \\ &\le C(n) \sum_{k > 2N}^{\infty} r^k (k+1)^{2n} \end{aligned}$$

where $C(n)$ is a constant independent of p and q. Therefore the series (1) converges and uniformly for $\eta, \zeta \in \partial B$ and $r \leq \rho < 1$. Fix $r < 1$. Let $f \in \mathcal{H}_s$. Then $f = \sum_{p+q=s} f_{pq}$ where $f_{pq} = \pi_{pq} f \in H(p,q)$.

$$\begin{aligned} P_m[f](z) &= \int_{\partial B} P_m(r\eta, \zeta) f(\zeta) d\sigma(\zeta) \\ &= \sum_{p+q=s} \int_{\partial B} P_m(r\eta, \zeta) f_{pq}(\zeta) d\sigma(\zeta). \end{aligned}$$

By the remarks in 3.3,

$$\begin{aligned} P_m[f](z) &= \sum_{p+q=s} C(p,q,m) R(p,q,m;r^2) f_{pq}(\eta) \\ &= \sum_{p+q=s} G_{p,q,m}(r) f_{pq}(\eta). \end{aligned} \tag{2}$$

Since

$$f_{pq} = (\pi_{pq} f)(\eta) = \int_{\partial B} K_{pq}(\eta, \zeta) f(\zeta) d\sigma(\zeta),$$

(2) has following form

$$\begin{aligned} P_m[f](z) &= \int_{\partial B} \sum_{p+q=s} G_{p,q,m} K_{pq}(\eta, \zeta) f(\zeta) d\sigma(\zeta) \\ &= \int_{\partial B} \sum_{p,q=0}^{\infty} G_{p,q,m} K_{pq}(\eta, \zeta) f(\zeta) d\sigma(\zeta) \end{aligned} \tag{3}$$

for $f \in \mathcal{H}_s$. Since the linear span of $\bigcup_{s=0}^{\infty} \mathcal{H}_s$ is dense $C(\partial B)$, (3) is true for any $f \in C(\partial B)$. Hence we have the expansion in (1).

4. Growth conditions

Two growth conditions are given, one for the Poisson type integral representations and the other for the projection of a function in Z_m onto $H(p,q)$ to be 0 unless either $p \leq m$ or $q \leq m$. We follow the arguments in [3] for the proofs.

4.1. Proposition . *$f = P_m[F]$ for some $F \in L^2(\partial B)$ if and only if $f \in Z_m$ and*

$$\sup_{0 \leq r < 1} \int_{\partial B} |f(r\zeta)|^2 d(\zeta) < \infty. \tag{1}$$

Proof. Suppose $f = P_m[F]$ and $F \in L^2(\partial B)$. Then $f \in Z_m$. We use the continous form of Minkowski inequality to get

$$\begin{aligned}
\int_{\partial B} |f(r\zeta)|^2 d\sigma(\zeta) &= \int_{\partial B} \left| \int_{\partial B} F(\eta) P_m(r\zeta,\eta) d\sigma(\eta) \right|^2 d\sigma(\zeta) \\
&\leq \int_{\partial B} \int_{\partial B} |F(\eta)|^2 P_m(r\zeta,\eta) d\sigma(\eta) d\sigma(\zeta) \\
&= \int_{\partial B} |F(\eta)|^2 \int_{\partial B} P_m(r\zeta,\eta) d\sigma(\zeta) d\sigma(\eta) \\
&= \alpha(n,m,r) \|F\|_2^2 \\
&\approx \|F\|_2^2.
\end{aligned}$$

Suppose $f \in Z_m$ and (1) holds. It follows from the remarks in 3.3 that

$$(\tilde{\pi}_{pq} f)(z) = R(p,q,m;|z|^2) f_{pq}(z), \qquad z \in B$$

for some $f_{pq} \in H(p,q)$. Since f is real-analytic in B, f lies in the closed linear span of $\tilde{\pi}_{pq} f$. Hence,

$$f(z) = \lim_{N \to \infty} \sum_{p+q \leq N} R(p,q,m;t) f_{pq}(z) \tag{2}$$

in the topology of uniform convergence on compact subsets of B. Define F by

$$F(\zeta) = \sum_{p,q} C(p,q,m)^{-1} f_{pq}(\zeta) \qquad (\zeta \in \partial B). \tag{3}$$

We will show that $F \in L^2(\partial B)$. From (1) and (2), we have

$$\begin{aligned}
\infty > C &\geq \int_{\partial B} |f(r\zeta)|^2 d\sigma(\zeta) \\
&= \sum_{p,q=0}^{\infty} R(p,q,m;r^2)^2 r^{2(p+q)} \|f_{pq}\|_2^2.
\end{aligned} \tag{4}$$

On the other hand,

$$R(p,q,m;r^2) = F(p-m, q-m, n+p+q; r^2)$$

which increases to

$$F(p-m, q-m, n+p+q; 1) \equiv= C(p,q,m)^{-1}$$

as $r \nearrow 1$. Therefore if we take limits as $r \nearrow 1$ in (4) we get

$$\sum C(p,q,m)^{-2} \|f_{pq}\|_2^2 < \infty. \tag{5}$$

(3) and (5) then imply $F \in L^2(\partial B)$. If we let

$$F_N(\zeta) = \sum_{p+q\leq N} C(p,q,m)^{-1} f_{pq}(\zeta) \qquad (\zeta \in \partial B) \tag{6}$$

and fix $z \in B, F_N \to F$ in $L^2(\partial B)$, so that

$$\lim_{N\to\infty} P_m[F_N](z) = P_m[F](z). \tag{7}$$

Therefore by (6),(7) and remarks in 3.3, we have

$$\begin{aligned}
P_m[F](z) &= \lim_{N\to\infty} P_m[F_N](z) \\
&= \lim_{N\to\infty} P_m[\sum_{p+q\leq N} C(p,q,m)^{-1} f_{pq}](z) \\
&= \lim_{N\to\infty} \sum_{p+q\leq N} C(p,q,m)^{-1} P_m[f_{pq}](z) \\
&= \lim_{N\to\infty} \sum_{p+q\leq N} C(p,q,m)^{-1} C(p,q,m) R(p,q,m;|z|^2) f_{pq}(z) \\
&= f(z).
\end{aligned}$$

4.2. Proposition . *Let $f \in Z_m$. If*

$$\int_{\partial B} |\mathcal{D}^{n+2m} f(r\zeta)|^2 d\sigma(\zeta) = o(\log^2 \frac{1}{1-r}) \tag{1}$$

as $r \to 1$ then $\pi_{pq} f = 0$ unless either $0 \leq p \leq m$ or $0 \leq q \leq m$.

Proof. Let $h(z) = \mathcal{D}^{n+2m} f(z)$. If we write

$$f(z) = \sum_{k=0}^{\infty} P_k(z,\bar{z}),$$

then $h(tz) = \sum_{k=0}^{\infty} (k+1)^{n+2m} t^k P_k(z,\bar{z})$. We note that

$$\begin{aligned}
&\frac{1}{\Gamma(n+2m)} \int_0^1 (\log \frac{1}{t})^{n+2m-1} h(tz) dt \\
&= \frac{1}{\Gamma(n+2m)} \sum_{k=0}^{\infty} (k+1)^{n+2m} \int_0^1 (\log \frac{1}{t})^{n+2m-1} t^k dt \\
&= \sum_{k=0}^{\infty} P_k(z,\bar{z}) \\
&= f(z).
\end{aligned}$$

Therefore, if

$$\int_{\partial B}\left\{\int_0^1 (1-t)^{n+2m-1}|h(tr\zeta)|dt\right\}^2 d\sigma(\zeta) \tag{2}$$

is bounded then

$$\int_{\partial B}|f(r\zeta|^2 d\sigma(\zeta)$$

is bounded. By Minkowski's inequality,

$$\begin{aligned}(2) \;\; &\leq \;\; \left[\int_0^1 (1-t)^{n+2m-1}\left\{\int_{\partial B}|h(tr\zeta)|^2 d\sigma(\zeta)\right\}^{\frac{1}{2}} dt\right]^2 \\ &\leq \;\; C(\int_0^1 (1-t)^{n+2m-1}\log\frac{1}{1-t}dt)^2 \qquad < \infty\end{aligned}$$

uniformly on r by (1) where C is a constant. By Proposition 4.1, there exists an $F \in L^2(\partial B)$ such that $f = P_m[F]$. From 2.4.(c),

$$F(\zeta) = \sum_{p,q} F_{pq}(\zeta)$$

in $L^2(\partial B)$ where $F_{pq} \in H(p,q)$. Let

$$F_N = \sum_{p+q\leq N} F_{pq}$$

and let $f_N = P_m[F_N]$. Then

$$f_N(z) = \sum_{p+q\leq N} C(p,q,m)R(p,q,m;|z|^2)F_{pq}(z) \qquad (z \in B).$$

Since $F_N \to F$ in $L^2(\partial B)$, we know that

$$\begin{aligned}&\mathcal{D}^{n+2m}f(r\eta) - \mathcal{D}^{n+2m}f_N(r\eta) \\ &= \int_{\partial B}\mathcal{D}^{n+2m}P_m(r\eta,\zeta)(F-F_N)(\zeta)d\sigma(\zeta) \\ &\longrightarrow 0 \text{ as } N \longrightarrow \infty.\end{aligned} \tag{3}$$

in $L^2(\partial B)$ once r is fixed.
Hence, by the orthogonality of F_{pq} and by (3), we have

$$\begin{aligned}&\int_{\partial B}|\mathcal{D}^{n+2m}f(r\zeta)|^2 d\sigma(\zeta) \\ &= \sum_{p,q}|C(p,q,m)|^2\|F_{pq}\|_2^2[\mathcal{D}^{n+2m}(R(p,q,m;r^2)r^{p+q})]^2.\end{aligned} \tag{4}$$

We note that

$$\begin{aligned}
&\mathcal{D}^{n+2m}(R(p,q,m;r^2)r^{p+q}) \\
&= \mathcal{D}^{n+2m}[F(p-m,q-m,n+p+q;r^2)r^{p+q}] \\
&= \sum_k \frac{(p-m)_k(q-m)_k}{(n+p+q)_k k!}(2k+p+q+1)^{n+2m} r^{2k+p+q}. \qquad (5)
\end{aligned}$$

Thus if both $p-m$ and $q-m$ are a positive integer then

$$\frac{(p-m)_k(q-m)_k}{(n+p+q)_k k!}(2k+p+q+1)^{n+2m} \approx \frac{1}{k}$$

as $k \to \infty$, so that $(5) > C(n,m,p,q)\log\left(\frac{1}{1-r}\right)$ for some positive constant $C(n,m,p,q)$. By (1),(4) and (5), $F_{pq} = 0$ unless either $p-m$ or $q-m$ is nonpositive integer.

Therefore $\pi_{pq}f = f_{pq} = P_m[F_{pq}] = 0$ unless either $p-m$ or $q-m$ is nonpositive integer.

Remark. If $m = 0$, above theorem corresponds to L^2-growth condition for a function g in X_λ to be in the $\mathcal{M}$-subspace Y_4 of X_λ. See [1,7].

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