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CONVEXITY OF THE PERMANENT FOR DOUBLY STOCHASTIC MATRICES II

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Dedicated to Professor Younki Chae on his sixtieth birthday

Let Ω_n denote the polytope consisting of all $n \times n$ doubly stochastic matrices. In this paper we prove the convexity of permanent for matrices in some subclasses of Ω_n.

1. Introduction

Let Ω_n denote the set of all $n \times n$ doubly stochastic matrices and let J_n denote the $n \times n$ matrix all of whose entries are $1/n$. It is very well known that Ω_n forms a convex polytope with all the $n!$ permutation matrices as its vertices.

For an $n \times n$ matrix $A = [a_{ij}]$, the *permanent* of A, per A, is defined by

$$\mathrm{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where S_n stands for the symmetric group on the set $\{1, 2, \cdots, n\}$. For a matrix $A \in \Omega_n$, let $f_A(t)$ denote the function of t defined by

$$f_A(t) := \mathrm{per}[(1-t)J_n + tA].$$

In 1978, Friendland and Minc [4] proved that for any $n \times n$ permutation matrix P, the function $f_p(t)$ is monotone increasing over the interval $0 \leq t \leq 1$ and is monotone decreasing over the interval $-1/(n-1) \leq t \leq 0$.

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If the function $f_A(t)$ is monotone increasing over the closed unit interval [0,1], then it is said that the *monotonicity of permanent* (abb. **MP**) holds for A. Friendland and Minc, *ibid.*, raised the problem of finding doubly stochasitc matices for which **MP** holds [4,8,13 *etc.*]. This property of the permanent function is referred to as the *monotonicity conjecture.* With the monotonicity conjecture still being open, several subclasss of Ω_n have been proved to satisfy **MP** [4,7,8,10,11,12,15,16].

Another particular property of the permanent function on Ω_n is the convexity which is as interesting as the monotonicity. In [5] we proposed the following.

Convexity Conjecture. Let $A \in \Omega_n$, $A \neq J_n$. Then the permanent function is strictly convex on the stright line segment joining J_n and $(J_n+A)/2$, and proved it for $n = 3$. We also conjectured [5] that the convexity conjecture and the following conjecture due to Lih and Wang are equivalent.

Lih and Wang's Conjecture [9]. Let $A \in \Omega_n$, then

$$\text{per}\ [(1-t)J_n + tA] \leq (1-t)\ \text{per}J_n + t\ \text{per}A$$

for all t in the closed interval $[0, 1/2]$.

In view of the convexity conjecture it will be of interest to find classes of doubly stochastic matrices for which the convexity of permanent over some subinterval of $[0,1]$ holds. In this paper, we prove that $f_p(t)$ is strictly convex on the interval $[-1/(n-1), 1]$ for all $n \times n$ permutation matrices P.

Throughout this paper, for an $n \times n$ matrix A and for subsets α, β of $\{1, \cdots, n\}$ let $A(\alpha \mid \beta)$ denote the matrix obtained from A by deleting rows lying in α and columns lying in β.

2. Preliminary Lemmas

Let n be a positive integer. For an integer k, $0 \leq k \leq n$, let $Q_{k,n}$ denote the set of all strictly increasing k-sequences from $\{1, 2, \cdots, n\}$. For an $n \times n$ matrix $A(t) = [a_{ij}(t)]$ whose entries are differentiable functions of the real variable t, and for $\alpha \in Q_{k,n}$, let $A^{(\alpha)}(t) = [b_{ij}(t)]$ and $A_{(\alpha)}(t) = [c_{ij}(t)]$ be defined by

$$b_{ij}(t) = \begin{cases} da_{ij}(t)/dt, & \text{if } i \in \alpha, \\ a_{ij}(t), & \text{otherwise,} \end{cases}$$

and

$$c_{ij}(t) = \begin{cases} da_{ij}(t)/dt, & \text{if } j \in \alpha, \\ a_{ij}(t), & \text{otherwise.} \end{cases}$$

Then we can easily prove the following lemma which gives a formula for $d\text{per}A(t)/dt$ similar to a well known formula for $d\det A(t)/dt$.

Lemma 1. *Let $A(t) = [a_{ij}(t)]$ be an $n \times n$ matrix whose entries are differentiable functions of t. Then*

$$\text{(1)} \qquad \frac{d}{dt}\text{per}A(t) = \sum_{i=1}^{n} \text{per}A^{(i)}(t).$$

$$\text{(2)} \qquad \frac{d}{dt}\text{per}A(t) = \sum_{j=1}^{n} \text{per}A_{(j)}(t).$$

From Lemma 1 there directly follows the following

Lemma 2. *Let $A(t)$ be the same as the one in Lemma 1. If all of the $a_{ij}(t)$'s are polynomials in t of degree 1 or less, then*

$$\text{(1)} \qquad \frac{d^k}{dt^k}\text{per}A(t) = k! \sum_{\alpha \in Q_{k,n}} \text{per}A^{(\alpha)}(t),$$

$$\text{(2)} \qquad \frac{d^k}{dt^k}\text{per}A(t) = k! \sum_{\alpha \in Q_{k,n}} \text{per}A_{(\alpha)}(t),$$

for all $k = 1, 2, \cdots, n$.

3. Convexity of the permanent for vertices of Ω_n

In this section we prove that the permanent function is convex on straight line segments joining J_n and vertices of Ω_n i.e. permutation matrices in Ω_n. It suffices to prove the above property for the identity matrix I_n only. For that purpose let

$$M_n(t) := (1-t)J_n + tI_n = [a_{ij}(t)].$$

Then

$$a_{ij}(t) = \begin{cases} \frac{1}{n} + (1 - \frac{1}{n})t, & \text{if } i = j, \\ \frac{1}{n} - \frac{1}{n}t, & \text{otherwise} \end{cases}$$

We are about to show that

$$\frac{d^2}{dt^2}\mathrm{per} M_n(t) \geq 0$$

over the interval $0 \leq t \leq 1$. Before starting this job, observe that per $M_n^{(\alpha)}(t)$ = per $M_n^{(1,2)}(t)$ for all $\alpha \in Q_{2,n}$, because for any $\alpha \in Q_{2,n}$ there are permutation matrices P, Q such that $PM_n^{(\alpha)}(t)Q = M_n^{(1,2)}(t)$.

Now we are ready to prove one of our main theorems.

Theorem 1. *The function* $f_{I_n}(t) = \mathrm{per} M_n(t)$ *is strictly convex over the interval* $0 \leq t \leq 1$.

Proof. By the above observation and by Lemma 2, we have

$$\begin{aligned} \frac{d^2}{dt^2}\mathrm{per} M_n(t) &= 2 \sum_{\alpha \in Q_{2,n}} \mathrm{per} M_n^{(\alpha)}(t) \\ &= n(n-1)\mathrm{per} M_n^{(1,2)}(t). \end{aligned}$$

Thus it suffices to show that per $M_n^{(1,2)}(t) > 0$ for all t, $0 < t < 1$. For t, $0 < t < 1$, let

$$x = 1 + \frac{nt}{1-t}$$

and let

$$(3.1)\quad U_n(x) = \left[\begin{array}{cc|ccc} 1-n & 1 & 1 & \cdots & 1 \\ & 1-n & 1 & \cdots & 1 \\ \hline 1 & 1 & & & \\ \vdots & \vdots & & (n-2)J_{n-2} + (x-1)I_{n-2} & \\ 1 & 1 & & & \end{array}\right].$$

Then since

$$M_n^{(1,2)}(t) = \frac{1}{n}\left[\begin{array}{cc|ccc} 1-n & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ \hline 1-t & 1-t & & & \\ \vdots & \vdots & & (n-2)(1-t)J_{n-2} + ntI_{n-2} & \\ 1-t & 1-t & & & \end{array}\right]$$

and

$$\frac{n-1}{-1} = 1-n, \quad \frac{1-t+nt}{1-t} = x,$$

we see that

$$\mathrm{per}M_n^{(1,2)}(t) = \frac{(1-t)^{n-2}}{n^n}\mathrm{per}U_n(x).$$

Hence our problem has been reduced to proving that per $U_n(x) > 0$ for all t, $0 < t < 1$, i.e., for all $x > 1$. For $n \leq 3$, it can be easily seen that per $U_n(x) > 0$ for all $x > 1$. So, assume $n \geq 4$.

Let E_{ij} denote the square matrix of suitable size all of whose entries are 0 except for the (i,j)-entry which is 1. For integers k, q with $0 \leq q < k$, let

$$C_{k,q} := kJ_k + (x-1)\sum_{i=1}^{k-q} E_{ii} \tag{3.2}$$

and $P(k,q) := \mathrm{per}C_{k,q}$.

Expanding per U_n along the first two rows, we get

$$\begin{aligned} \mathrm{per}U_n &= (n^2-2n+2)P(n-2,0) - 2(n-2)^2P(n-2,1) \\ &\quad +(n^2-5n+6)P(n-2,2). \end{aligned} \tag{3.3}$$

Since $C_{n-2,0} = C_{n-2,1} + (x-1)E_{n-2,n-2}$, we get

$$\begin{aligned} P(n-2,0) &= \mathrm{per}C_{n-2,1} + (x-1)\mathrm{per}C_{n-2,1}(n-2 \mid n-2) \\ &= P(n-2,1) + (x-1)P(n-3,0) \end{aligned} \tag{3.4}$$

since $C_{n-2,1}(n-2 \mid n-2) = C_{n-3,0}$.

Similarly we can show that

$$P(n-2,1) = P(n-2,2) + (x-1)P(n-3,1). \tag{3.5}$$

Hence

$$\begin{aligned} P(n-2,0) &= P(n-2,2) + (x-1)P(n-3,1) \\ &\quad +(x-1)P(n-3,0) \\ &\geq P(n-2,2) + 2(x-1)P(n-3,1) \end{aligned} \tag{3.6}$$

since $P(n-3,0) \geq P(n-3,1)$. Now (3.3), (3.5) and (3.6) yield

$$\mathrm{per}U_n \geq nP(n-2,2) + 4(n-1)(x-1)P(n-3,1) > 0$$

since $x \geq 1$, and the proof is complete.

We now prove the convexity of the permanent on the straight line segment joining J_n and $(nJ_n - P)/(n-1)$ for $n \times n$ permutation matrices P. For this purpose, let

$$T_n := \frac{nJ_n - I_n}{n-1}$$

and let $S_n(t) := (1-t)J_n + tT_n = [s_{ij}(t)]$. Then

$$s_{ij}(t) = \begin{cases} \frac{1-t}{n}, & \text{if } i = j, \\ \frac{1-t}{n} + \frac{t}{n-1}, & \text{otherwise.} \end{cases}$$

Whence

$$S_n^{(1,2)}(t) = \frac{1}{n(n-1)} \left[\begin{array}{cc|ccc} 1-n & 1 & 1 & \cdots & 1 \\ 1 & 1-n & 1 & \cdots & 1 \\ \hline n-1+t & n-1+t & & & \\ \vdots & \vdots & & (n-2)(n-1+t)J_{n-2} - ntI_{n-2} & \\ n-1+t & n-1+t & & & \end{array} \right],$$

so that

$$\text{per}S_n^{(1,2)}(t) = \frac{(n-1+t)^{n-2}}{n^n(n-1)^n} \text{per}U_n(x)$$

where $U_n(x)$ is the matrix (2.1) with

$$x = \frac{(n-1)(1-t)}{n-1+t}.$$

Notice in this case that $0 \le x \le 1$ while t ranges over the interval $0 \le t \le 1$.

As in the case of $M_n(t)$, we see that per $S_n^{(\alpha)}(t) = \text{per}S_n^{(1,2)}(t)$ for any 2-subset α of $\{1, 2, \cdots, n\}$, and hence also that

$$\frac{d^2}{dt^2}\text{per}S_n(t) = \frac{n(n-2)(n-1+t)^{(n-2)}}{n^n(n-1)^n}\text{per}U_n(x). \tag{3.7}$$

With this in mind we now prove the following

Theorem 2. *The permanent function is strictly convex on the stright line segment joining J_n and $T_n = (nJ_n - I_n)/(n-1)$, for all $n \ge 4$.*

Proof. By the equality (3.7) we need only to show that per $U_n(x) > 0$ for all x with $0 < x < 1$.

For the respective cases $n = 4$ and $n = 5$ we can show that

$$\begin{aligned} \mathrm{per}U_4(x) &= 10x^2 - 8x + 6 > 0, \text{ for } \frac{-1}{3} < x < 1, \\ \mathrm{per}U_5(x) &= 17x^3 - 18x^2 + 27x + 4 > 0, \text{ for } \frac{-1}{4} < x < 1. \end{aligned}$$

Suppose that $n \geq 6$. Then by (3.4), (3.5) and (3.6), we have

$$\begin{aligned} (3.8) \quad \mathrm{per}U_n &= (n^2 - 2n + 2)[P(n-2,2) + (x-1)\{P(n-3,1) \\ &\quad + P(n-3,0)\}] \\ &\quad -2(n-2)^2\{P(n-2,2) + (x-1)P(n-3,1)\} \\ &\quad +(n^2 - 5n + 6)P(n-2,2). \end{aligned}$$

Since $P(n-3,0) \geq P(n-3,1)$ because $0 \leq x \leq 1$, we have

$$\begin{aligned} (3.9) \quad \mathrm{per}U_n &\geq nP(n-2,2) - \{2(n^2 - 2n + 2) + 2(n-2)^2\} \\ (3.10) \quad &\qquad (1-x)P(n-3,1) \\ &\geq nP(n-2,2) - 4(n-1)P(n-3,1) \end{aligned}$$

for all x such that $0 \leq x \leq 1$.

Let $C_{k,q}$ be the matrix defined by (3.2) with $0 \leq x \leq 1$. Expanding per $C_{n-2,2}$ along the last column, we have

$$(3.11) \qquad P(n-2,2) = 2P(n-3,1) + (n-4)P(n-3,2).$$

From (3.9) and (3.10), it follows that

$$\begin{aligned} \mathrm{per}U_n &\geq (-2n+4)P(n-3,1) + n(n-4)P(n-3,0) \\ &\geq (n^2 - 6n + 4)P(n-3,1), \end{aligned}$$

which is positive for $n \geq 6$ and the proof is complete.

For $n = 3$, it is not true that the function $f_{T_3}(t) = \mathrm{per}[(1-t)J_3 + tT_3]$ is convex on the entire unit interval [0,1]. In fact $f_{T_3}(t)$ is strictly convex on $[0, 1/2)$ and strictly concave on $(1/2, 1]$.

Our Theorems 1 and 2 can be combined as follows.

Theorem 3. *For $n \geq 4$, the function $f_P(t) = \mathrm{per}[(1-t)J_n + tP]$ is strictly convex on the interval $-1/(n-1) \leq t \leq 1$, for any $n \times n$ permutation matrix P.*

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