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PARTIAL ALGEBRAS OVER TOPOLOGICAL CONSTRUCTS

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Dedicated to Professor Younki Chae and Professor Tae Ho Choe on their 60th birthday

0. Introduction

For a categorical setting for algebraic structures, algebraic functors, essentially algebraic functors and monadic functors have been introduced (see [1]) but the underlying set functor U on the category $\underline{\text{PAlg}}_\tau$ of partial algebras of type τ and homomorphisms into the category $\underline{\text{Set}}$ of sets and maps is not essentially algebraic ([2]), because it does not create isomorphisms.

In this paper, we introduce the category $\underline{\text{Rel}}_\tau$ of τ-relational sets and relation-preserving maps, which is a topological construct and then for any topological construct $\underline{\text{X}}$, we have the mixed category $\underline{\text{Rel}}_\tau(\underline{\text{X}})$ of $\underline{\text{Rel}}_\tau$ and $\underline{\text{X}}$, which is also a topological construct ([5]). Furthermore, we form the category $\underline{\text{PAlg}}_\tau(\underline{\text{X}})$ of partial algebras of type τ in $\underline{\text{X}}$ and show that the forgetful functor $\text{G} : \underline{\text{PAlg}}_\tau(\underline{\text{X}}) \longrightarrow \underline{\text{Rel}}_\tau(\underline{\text{X}})$ is essentially algebraic. Thus instead of $\underline{\text{Set}}$, $\underline{\text{Rel}}_\tau$ and $\underline{\text{Rel}}_\tau(\underline{\text{X}})$ are proper base categories for the categorical setting of partial algebras and topological partial algebras, respectively.

For the categorical terminologies, we refer to [1] and for those of algebras, we refer to [3] and [6].

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1. Categories $\underline{\mathrm{Rel}}_\tau(\underline{\mathrm{X}})$

In the following, $\underline{\mathrm{X}}$ will always denote a topological construct like the category $\underline{\mathrm{Top}}$ ($\underline{\mathrm{Unif}}$, $\underline{\mathrm{Conv}}$, $\underline{\mathrm{Qord}}$) of topological (uniform, convergence, quasi ordered, resp.) spaces and their corresponding morphisms, and τ will denote a type $(\lambda_\mu)_{\mu\in M}$ of algebras, i.e., a family $\tau = (\lambda_\mu)_{\mu\in M}$ of ordinals indexed by a set M.

Now we define τ-relational sets as follows:

Definition 1.1. 1) A pair $(\mathrm{X}, (\mathrm{X}_\mu)_{\mu\in M})$ is said to be a *τ-relational set* if X is a set and for any $\mu \in M$, $\mathrm{X}_\mu \subseteq \mathrm{X}^{\lambda_\mu}$, i.e., X_μ is a λ_μ-ary relation on the set X.

2) For τ-relational sets $(\mathrm{X}, (\mathrm{X}_\mu)_{\mu\in M})$, $(\mathrm{Y}, (\mathrm{Y}_\mu)_{\mu\in M})$, a map $f : \mathrm{X} \longrightarrow \mathrm{Y}$ is said to be a *relation preserving map* if for any $\mu \in M$, $f^{\lambda_\mu}(\mathrm{X}_\mu) \subseteq \mathrm{Y}_\mu$, where $f^{\lambda_\mu} : \mathrm{X}^{\lambda_\mu} \longrightarrow \mathrm{Y}^{\lambda_\mu}$ denotes the λ_μ-th power of f.

It is clear that the class of τ-relational sets and relation preserving maps forms a category, which will be denoted by $\underline{\mathrm{Rel}}_\tau$ and that for $\tau = \{2\}$, $\underline{\mathrm{Rel}}_\tau$ is precisely the category $\underline{\mathrm{Rel}}$.

Let $\mathrm{R} : \underline{\mathrm{Rel}}_\tau \longrightarrow \underline{\mathrm{Set}}$ denote the underlying set functor. Then for any source $(f_i : \mathrm{X} \longrightarrow \mathrm{R}((\mathrm{X}_i, (\mathrm{X}_{i\mu})_{\mu\in M})))_{i\in I}$, let $\mathrm{X}_\mu = \cap\{(f_i^{\lambda_\mu})^{-1}(\mathrm{X}_{i\mu}) \colon i \in \mathrm{I}\}$, then one can easily show that the source $(f_i : (\mathrm{X}, (\mathrm{X}_\mu)_{\mu\in M}) \longrightarrow (\mathrm{X}_i, (\mathrm{X}_{i\mu})_{\mu\in M}))_I$ is the unique R-initial lift of $(f_i)_I$. Thus we have:

Theorem 1.2. *The category $\underline{\mathrm{Rel}}_\tau$ is a topological construct.*

Remark 1.3. 1) A sink $(f_i : (\mathrm{X}_i, (\mathrm{X}_{i\mu})_{\mu\in M}) \longrightarrow (\mathrm{X}, (\mathrm{X}_\mu)_{\mu\in M}))_{i\in I}$ is an R-final sink iff for any $\mu \in M$, $\mathrm{X}_\mu = \cup\{f_i^{\lambda_\mu}(\mathrm{X}_{i\mu}) \colon i \in \mathrm{I}\}$. In particular, a morphism $f : (\mathrm{X}, (\mathrm{X}_\mu)_{\mu\in M}) \longrightarrow (\mathrm{Y}, (\mathrm{Y}_\mu)_{\mu\in M})$ in $\underline{\mathrm{Rel}}_\tau$ is a quotient morphism iff it is onto and for all $\mu \in M$, $f^{\lambda_\mu}(\mathrm{X}_\mu) = \mathrm{Y}_\mu$.

2) An inclusion map $j : (\mathrm{X}, (\mathrm{X}_\mu)_{\mu\in M}) \longrightarrow (\mathrm{Y}, (\mathrm{Y}_\mu)_{\mu\in M})$ is R-initial iff for all $\mu \in M$, $\mathrm{X}_\mu = \mathrm{X}^{\lambda_\mu} \cap \mathrm{Y}_\mu$.

3) For any $\mathrm{X} \in \underline{\mathrm{Set}}$, $(\mathrm{X}, (\emptyset)_{\mu\in M})$ is a discrete object and $(\mathrm{X}, (\mathrm{X}^{\lambda_\mu})_{\mu\in M})$ is an indiscrete object in $\underline{\mathrm{Rel}}_\tau$, which will give rise to the left adjoint and the right adjoint of R, respectively.

Definition 1.4. A τ-relational set $(\mathrm{X}, (\mathrm{X}_\mu)_{\mu\in M})$ is said to be *reflexive* if each X_μ contains the λ_μ-diagonal relation in X.

Let $\underline{\mathrm{dRel}}_\tau$ denote the full subcategory of $\underline{\mathrm{Rel}}_\tau$ determined by the reflexive τ-relational sets. By adding the corresponding diagonal relation to each relation in an object $(X, (X_\mu)_{\mu\in M})$ in $\underline{\mathrm{Rel}}_\tau$, one can show that $\underline{\mathrm{dRel}}_\tau$ is a bireflective subcategory of $\underline{\mathrm{Rel}}_\tau$; therefore $\underline{\mathrm{dRel}}_\tau$ is also a topological construct. Furthermore, for any $(X, (X_\mu)_{\mu\in M})$ and $(Y, (Y_\mu)_{\mu\in M})$ in $\underline{\mathrm{dRel}}_\tau$, let $Y^X = \hom((X, (X_\mu)_{\mu\in M}), (Y, (Y_\mu)_{\mu\in M}))$ and for any $\mu \in M$, let $F_\mu = \{(f_i) \in (Y^X)^{\lambda_\mu} : (\prod f_i)(X_\mu) \subseteq Y_\mu\}$. Then by a routine calculation, the evaluation map ev: $X\times Y^X \longrightarrow Y$ is indeed the couniversal map for the functor $X\times\_ : \underline{\mathrm{dRel}}_\tau \longrightarrow \underline{\mathrm{dRel}}_\tau$. Hence one has the following:

Theorem 1.5. *The category $\underline{dRel}_\tau$ is a cartesian closed topological construct.*

For a topological construct $\underline{X}$, let $\underline{\mathrm{Rel}}_\tau(\underline{X})$ denote the mixed category of $\underline{\mathrm{Rel}}_\tau$ and $\underline{X}$ (see [5] for the detail), then the following is immediate from the result in [5].

Theorem 1.6. 1) *The category $\underline{Rel}_\tau(\underline{X})$ is a topological construct.*

2) *If X is cartesian closed, then the category $\underline{dRel}_\tau(\underline{X})$ is also cartesian closed.*

2. Partial Algebras over a Topological Construct $\underline{X}$

In this section, we introduce the category $\underline{\mathrm{PAlg}}_\tau(\underline{X})$ of partial algebras of type τ over a topological construct $\underline{X}$ and their $\underline{X}$-homomorphisms and then show that the forgetful functor $G : \underline{\mathrm{PAlg}}_\tau(\underline{X}) \longrightarrow \underline{\mathrm{Rel}}_\tau(\underline{X})$ is essentially algebraic.

Definition 2.1. Let $\underline{X}$ be a topological construct and $\tau = (\lambda_\mu)_{\mu\in M}$ a type of algebras.

1) A pair $A = (X, (f_\mu)_{\mu\in M})$ is said to be an *$\underline{X}$-partial algebra of type τ* or simply *$\underline{X}$-partial algebra* if X is an object of $\underline{\mathrm{Rel}}_\tau(\underline{X})$ and for each $\mu \in M$, $f_\mu : X_\mu \longrightarrow X$ is an $\underline{X}$-morphism, where X_μ is the μ-th relation on X. In this case, f_μ is called a λ_μ-ary partial operation on A.

2) For $\underline{X}$-partial algebras $A = (X, (f_\mu)_{\mu\in M})$, $B = (Y, (g_\mu)_{\mu\in M})$ of type τ, a $\underline{\mathrm{Rel}}_\tau(\underline{X})$-morphism $h : X \longrightarrow Y$ is said to be a *homomorphism* on A to B if for any $\mu \in M$, $h \circ f_\mu = g_\mu \circ h^{\lambda_\mu}$, where h^{λ_μ} denotes the restriction and corestriction of the λ_μ-th power of h to X_μ and Y_μ, respectively.

We form the category of $\underline{X}$-partial algebras of type τ and their homomorphims, which will be denoted by $\underline{\mathrm{PAlg}}_\tau(\underline{X})$. Moreover $\underline{\mathrm{PAlg}}_\tau(\underline{\mathrm{Set}})$ will be simply denoted by $\underline{\mathrm{PAlg}}_\tau$.

Since $\underline{\mathrm{PAlg}}_\tau$ is mono-topological, the underlying set functor U on $\underline{\mathrm{PAlg}}_\tau$ into $\underline{\mathrm{Set}}$ has a left adjoint and hence the forgetful functor $G_1 : \underline{\mathrm{PAlg}}_\tau \longrightarrow$

$\underline{\mathrm{Rel}}_\tau$ has also a left adjoint. Here we give another proof as follows.

Lemma 2.2. *The forgetful functor G_1: $\underline{PAlg}_\tau \longrightarrow \underline{Rel}_\tau$ has a left adjoint.*

Proof. Since every mono-source in $\underline{\mathrm{PAlg}}_\tau$ is clearly G_1-initial, $\underline{\mathrm{PAlg}}_\tau$ is a complete category and G_1 preserves limits. Thus it is enough to show that every object in $\underline{\mathrm{Rel}}_\tau$ has a G_1-solution set. Take any object $\mathrm{X} = (\mathrm{X}, (\mathrm{X}_\mu)_{\mu\in M})$ in $\underline{\mathrm{Rel}}_\tau$. We may assume that X is non-empty, for if X is empty, X endowed with empty operations will give rise to a G_1-solution set for X. Let $(\mathrm{F}(\mathrm{X}), (f_\mu)_{\mu\in M})$ be the absolutely free algebra of type τ generated by X and let $g_\mu = f_\mu|\mathrm{X}_\mu$. Then the inclusion map $j : (\mathrm{X}, (\mathrm{X}_\mu)_{\mu\in M}) \longrightarrow \mathrm{G}_1((\mathrm{F}(\mathrm{X}), (\mathrm{X}_\mu)_{\mu\in M}, (g_\mu)_{\mu\in M}))$ is a G_1-solution set for X, because for any $f : \mathrm{X} \longrightarrow \mathrm{G}_1(\mathrm{A})$ in $\underline{\mathrm{Rel}}_\tau$, each partial operation on A can be extended to a full operation on A; hence f can be uniquely extended to a full homomorphism on F(X) to the full algebra A. The detail of the proof is left to the readers.

In the following, let $\mathrm{G} : \underline{\mathrm{PAlg}}_\tau(\underline{\mathrm{X}}) \longrightarrow \underline{\mathrm{Rel}}_\tau(\underline{\mathrm{X}})$, $\mathrm{V} : \underline{\mathrm{Rel}}_\tau(\underline{\mathrm{X}}) \longrightarrow \underline{\mathrm{X}}$, $\mathrm{U}_1 : \underline{\mathrm{PAlg}}_\tau(\underline{\mathrm{X}}) \longrightarrow \underline{\mathrm{PAlg}}_\tau$, $\mathrm{U}_2 : \underline{\mathrm{Rel}}_\tau(\underline{\mathrm{X}}) \longrightarrow \underline{\mathrm{Rel}}_\tau$, $\mathrm{U}_3 : \underline{\mathrm{X}} \longrightarrow \underline{\mathrm{Set}}$, $\mathrm{G}_1 : \underline{\mathrm{PAlg}}_\tau \longrightarrow \underline{\mathrm{Rel}}_\tau$, $\mathrm{R} : \underline{\mathrm{Rel}}_\tau \longrightarrow \underline{\mathrm{Set}}$ and $\mathrm{U} : \underline{\mathrm{PAlg}}_\tau\ (\underline{\mathrm{X}}) \longrightarrow \underline{\mathrm{Set}}$ denote the forgetful functors, then $\mathrm{U}_2\circ\mathrm{G} = \mathrm{G}_1\circ\mathrm{U}_1$, $\mathrm{U}_2\circ\mathrm{R} = \mathrm{V}\circ\mathrm{U}_3$ and $\mathrm{U} = \mathrm{U}_3\circ\mathrm{V}\circ\mathrm{G}$. Moreover, for any source $(f_i : \mathrm{A} \longrightarrow \mathrm{U}_1(\mathrm{A}_i))_{i\in I}$ in $\underline{\mathrm{PAlg}}_\tau$, the object A endowed with the U_3-initial $\underline{\mathrm{X}}$-structure with respect to $(f_i)_I$ is again an object of $\underline{\mathrm{PAlg}}_\tau(\underline{\mathrm{X}})$, which gives rise to the U_1-initial lift of $(f_i)_I$. Using this and the above lemma, one has the following:

Theorem 2.3. *The functor G: $\underline{PAlg}_\tau(\underline{X}) \longrightarrow \underline{Rel}_\tau(\underline{X})$ has a left adjoint.*

Proof. Take any $X = (X, (X_\mu)_{\mu\in M})$ in $\underline{\mathrm{Rel}}_\tau(\underline{\mathrm{X}})$. Let $\eta : \mathrm{U}_2(\mathrm{X}) \longrightarrow \mathrm{G}_1(\mathrm{A})$ be the G_1-universal map for $\mathrm{U}_2(\mathrm{X})$ and let $(h_i : \mathrm{X} \longrightarrow \mathrm{G}(\mathrm{A}_i))_{i\in I}$ be the source of all $\underline{\mathrm{Rel}}_\tau(\underline{\mathrm{X}})$-morphisms, then for any $i \in \mathrm{I}$, there is a unique $\underline{\mathrm{PAlg}}_\tau$-morphism $k_i : \mathrm{A} \longrightarrow \mathrm{A}_i$ with $k_i \circ \eta = h_i$. Let B be the $\underline{\mathrm{PAlg}}_\tau(\underline{\mathrm{X}})$-object A endowed with the U_3-initial $\underline{\mathrm{X}}$-structure with respect to $(k_i)_I$, then it is straightforward to show that $\eta : \mathrm{X} \longrightarrow \mathrm{G}(\mathrm{B})$ is a G-universal map for X. This completes the proof.

The following definition is due to Herrlich [4].

Definition 2.4. A functor $\mathrm{G} : \mathrm{A} \longrightarrow \mathrm{B}$ is said to be *essentially algebraic* if it creates isomorphisms and is (G-epi, Mono-Source)-factorizable.

It is known that essentially algebraic functors have rich categorical properties (see Chapter 23 in [1]).

Lemma 2.5. *The category $\underline{PAlg}_\tau(\underline{X})$ is (Epi, Mono-Source)-factorizable.*
Proof. Let $(f_i : A \longrightarrow A_i)_{i\in I}$ be a source in $\underline{PAlg}_\tau(\underline{X})$. Considering the intersection of the family $\{\ker(f_i) : i \in I\}$, one can easily show that $\underline{PAlg}_\tau$ is (Epi, Mono-Source)-factorizable. Indeed, let $\theta = \cap\{\ker(f_i) : i \in I\}$, then θ is a congruence of A and let $q : A \longrightarrow A/\theta$ be the quotient homomorphism, which is G_1-final. Thus there is a unique morphism $m_i : A/\theta \longrightarrow A_i$ in $\underline{PAlg}_\tau$ with $m_i \circ q = f_i$, and $(m_i)_I$ is a mono-source, which is also G_1-initial. Let E denote the $\underline{PAlg}_\tau(\underline{X})$-object A/θ endowed with the U_3-initial $\underline{X}$-structure with respect to $(m_i)_I$. Then it is clear that $q : A \longrightarrow E$ and each $m_i : E \longrightarrow A_i$ are $\underline{PAlg}_\tau(\underline{X})$-morphisms and that $f_i = m_i \circ q$ $(i \in I)$ is an (Epi, Mono-Source)-factorization.

Theorem 2.6 *The functor $G : \underline{PAlg}_\tau(\underline{X}) \longrightarrow \underline{Rel}_\tau(\underline{X})$ is essentially algebraic.*

Proof. By Theorem 2.3 and Lemma 2.5, it is enough to show that G creates isomorphisms (see Theorem 23.8 in [1]). Take any isomorphism $h : X \longrightarrow G(A)$ in $\underline{Rel}_\tau(\underline{X})$, i.e., $h : X \longrightarrow G(A)$ is both an $\underline{X}$-isomorphism and a $\underline{Rel}_\tau(\underline{X})$-isomorphism, where $X = (X, (X_\mu)_{\mu\in M})$ and $A = (A, (A_\mu)_{\mu\in M}, (f_\mu)_{\mu\in M})$. For any $\mu \in M$, let $g_\mu = h^{-1} \circ f_\mu \circ h^{\lambda_\mu} : X_\mu \longrightarrow X$, where h^{λ_μ} denotes the restriction and corestriction of the λ_μ-th power of h to X_μ and A_μ , respectively as before. Then it is clear that each g_μ is an $\underline{X}$-morphism; hence $(X, (X_\mu)_{\mu\in M}, (g_\mu)_{\mu\in M})$ is a $\underline{PAlg}_\tau(\underline{X})$-object and that $h : (X, (X_\mu)_{\mu\in M}, (g_\mu)_{\mu\in M}) \longrightarrow (A, (A_\mu)_{\mu\in M}, (f_\mu)_{\mu\in M})$ is an isomorphism in $\underline{PAlg}_\tau(\underline{X})$ with $G(h) = h$. This completes the proof.

The following is now immediate from the above theorem and results in [1].

Corollary 2.7. 1) *G detects colimits and preserves and creates limits.*
2) *$\underline{PAlg}_\tau(\underline{X})$ is complete and cocomplete.*
3) *The functor $G_1 : \underline{PAlg}_\tau \longrightarrow \underline{Rel}_\tau$ is essentially algebraic.*

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