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CONVERGENCE OF WEIGHTED AVERAGES OF MARTINGALE DIFFERENCE SEQUENCE

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Dedicated to Professor Younki Chae on his 60th birthday

Introduction

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, and $(\mathcal{F}_n)$ an increasing sequence of sub-σ-algebras of $\mathcal{F}$. For $X \in L^1(= L^1(P))$, we use $E_n(X)$ to denote $E(X|\mathcal{F}_n)$, the conditional expectation of X given $\mathcal{F}_n$. A sequence $X_n \in L^1(\mathcal{F}_n)$ will be called a martingale difference sequence if $E_n(X_{n+1}) = 0, n \in N$ (N is the set of positive integers). Throughout this paper $\{X_k\}$ will be an identically distributed martingale difference sequence and $\{w_k\}$ a sequence of positive numbers and write $S_n = \sum_{k=1}^n w_k X_k$. $\{B_n\}$ is a monotone sequence of positive constants with $B_n \to \infty$ as $n \to \infty$. In this paper we shall be concerned with the questions of almost sure convergence and L^1-convergence of $B_n^{-1} S_n$, and integrability of maximal function of the average $M(\omega) = \sup_n B_n^{-1}|S_n(\omega)|$.

In the special case where $w_k = 1$, all k, Elton [3] proved that if $E|X_1|log^+|X_1| < \infty$, then $S_n/n \to 0$ almost surely and $EM < \infty$. Also he showed that if $E|X| < \infty$ with $EX = 0$ but $E|X|log^+|X| = \infty$, there exists an identically distributed martingale difference sequence $\{X_n\}$ with X_1 having the same distribution as X such that S_n/n diverses almost surely. In the case where the $\{X_k\}$ being independent and identically distributed, many mathematicians have studied but this paper is related with Heyde [4] and Jamison, Orey, and Pruitt [5]. Our results provide a generalization of some work of Elton [3].

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Results

For a given sequence of positive weights $\{w_k\}$, define for each $x > 0$ $N(x)$ as the number of n such that $B_n/w_n \le x$. The rate of growth of this function is the critical factor in establishing almost sure convergence. Let F be the distribution function of random variable X_1.

Theorem 1. *If both $\int x^2 \int_{y\ge|x|} \frac{N(y)}{y^3} dydF(x) < \infty$ and $\int \ |x| \int_{0\le y\le|x|} \frac{N(y)}{y^2} dydF(x) < \infty$, then $B_n^{-1}S_n \to 0$ almost surely as $n \to \infty$.*

Proof. Let Y_k be X_k truncated at B_k/w_k and $T_n = \sum_1^n w_kY_k$. Put $Z_k = X_k - Y_k$ and $d_k = Y_k - E_{k-1}(Y_k)$. Write

$$\begin{aligned} X_n &= Y_n + Z_n = (Y_n - E_{n-1}(Y_n)) + E_{n-1}(Y_n) + Z_n \\ &= d_n + (Z_n - E_{n-1}(Z_n)), \end{aligned}$$

observing that $E_{n-1}(Z_n) = -E_{n-1}(Y_n)$ since $E_{n-1}(X_n) = 0$. Then

$$\begin{aligned} E(|\sum_{k=1}^{n}(w_k/B_k)X_k|) &\le E(|\sum_{k=1}^{n}(w_k/B_k)d_k|) + 2\sum_{k=1}^{n}(w_k/B_k)E|Z_k| \\ &\le (\sum_{k=1}^{n}(w_k/B_k)^2E(d_k^2))^{1/2} + 2\sum_{k=1}^{n}(w_k/B_k)E|Z_k|, \end{aligned}$$

since the d_k are orthogonal elements of L^2, and E_{k-1} is a contraction on L^1.

Next, Lemma 1 and 2, which follows, show that

$$\sum_{n=1}^{\infty}(w_n/B_n)^2E(d_n^2) < \infty \text{ and } \sum_{n=1}^{\infty}(w_n/B_n)E|Z_n| < \infty,$$

so $B_n^{-1}S_n \to 0$ almost surely as $n \to \infty$ by Kronecker's lemma, and the proof is complete.

Lemma 1. *If $\int x^2 \int_{y\ge|x|} \frac{N(y)}{y^3} dydF(x) < \infty$, then*

$$\sum_{n=1}^{\infty}(w_n^2/B_n^2)E(d_n^2) < \infty.$$

Proof. First note that $I - E_{n-1}$ is a contraction on L^2, so

$$E(d_n^2) = E((Y_n - E_{n-1}(Y_n))^2) \le E(Y_n^2).$$

Now

$$\sum_{k=1}^{\infty} \frac{w_k^2 E Y_k^2}{B_k^2} = \sum_{k=1}^{\infty} \frac{w_k^2}{B_k^2} \int_{|x|<(B_k/w_k)} x^2 \, dF(x) = \int x^2 \sum_{\{k:|x|<(B_k/w_k)\}} \frac{w_k^2}{B_k^2} \, dF(x),$$

and

$$\sum_{\{k:|x|<(B_k/w_k)\le z\}} \frac{w_k^2}{B_k^2} = \int_{|x|<y\le z} \frac{dN(y)}{y^2} = \frac{N(z)}{z^2} - \frac{N(|x|)}{x^2} + 2\int_{|x|<y\le z} \frac{N(y)}{y^3} dy,$$

while

$$\frac{N(z)}{z^2} \le 2\int_{z\le y} N(y)/y^3 dy \to 0 \ \ as \ \ z \to \infty,$$

the integral converging since $\int x^2 \int_{y\ge|x|} \frac{N(y)}{y^3} \, dydF(x) < \infty$, so that

$$\sum_{\{k:|x|<(B_k/w_k)\}} \frac{w_k^2}{B_k^2} \le 2\int_{y\ge|x|} \frac{N(y)}{y^3} \, dy,$$

and hence

$$\sum_{k=1}^{\infty} \frac{w_k^2 E(d_k^2)}{B_k^2} \le 2\int x^2 \int_{y\ge|x|} \frac{N(y)}{y^3} \, dydF(x) < \infty.$$

Lemma 2. *If both* $\int |x| \int_{y\le|x|} \frac{N(y)}{y^2} \, dydF(x) < \infty$ *and* $\int x^2 \int_{y\ge|x|} \frac{N(y)}{y^3}$ $dydF(x) < \infty$*, then*

$$\sum_{n=1}^{\infty} (w_n/B_n) E|Z_n|) < \infty.$$

Proof.

$$\sum_{n=1}^{\infty} \frac{w_n E|Z_n|}{B_n} = \sum_{n=1}^{\infty} \frac{w_n}{B_n} \int_{|x|\ge(B_n/w_n)} |x| \, dF(x) = \int [|x| \sum_{\{n:|x|\ge(B_n>w_n)\}} \frac{w_n}{B_n}] \, dF(x)$$

and

$$\sum_{\{n:|x|\geq(B_n/w_n)>z\}} \frac{w_n}{B_n} = \frac{N(|x|)}{x} - \frac{N(z)}{z} + \int_{|x|\geq y>z} \frac{N(y)}{y^2} dy,$$

so that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{w_n E|Z_n|}{B_n} &\leq \int N(|x|)dF(x) + \int |x| \int_{y\leq|x|} \frac{N(y)}{y^2} dy dF(x) \\ &= EN(|X_1|) + \int |x| \int_{y\leq|x|} \frac{N(y)}{y^2} dy dF(x) < \infty,\end{aligned}$$

observing that if $\int x^2 \int_{y\geq|x|} \frac{N(y)}{y^3} dydF(x) < \infty$,then $EN(|X_1|) < \infty$.

The following theorem is a generalization of Theorem 2 [3] with $w_k = 1$, for all k and $B_n = n$ for all n.

Theorem 2. *If $E|X_1|log^+|X_1| < \infty$ and lim sup $N(x)/x < \infty$ as $n \to \infty$, then $S_n/B_n \to 0$ almost surely.*

Proof. Suppose that lim sup $N(x)/x < \infty$ so that $N(x) < Kx$ for all $x > 0$. Then

$$\int x^2 \int_{y\geq|x|} \frac{N(y)}{y^3} \, dydF(x) \leq \int x^2 \frac{K}{|x|} dF(x) = KE|X_1| < \infty,$$

and

$$\begin{aligned} intx \int_{0\leq y\leq|x|} \frac{N(y)}{y^2} dydF(x) &\leq K \int |x| log^+|x| \, dF(x) \\ &= KE|X_1|log^+|X_1| < \infty \end{aligned}$$

so the strong law applies by Theorem 1.

Theorem 3. *Under the conditions of Theorem 1 we have $M \in L^1$.*

Proof. Write

$$X_n = d_n + (Z_n - E_{n-1}(Z_n)).$$

Since for any sequence of real numbers $\{a_n\}$

$$\begin{aligned} &|(1/B_n)\textstyle\sum_{k=1}^n w_k a_k| = |\textstyle\sum_{k=1}^n (1/B_k) w_k a_k (1 - \dfrac{B_n - B_k}{B_n})| \\ &\leq |\sum_{k=1}^n (w_k/B_k) a_k| + |\textstyle\sum_{k=1}^n (w_k/B_k) a_k (B_n - B_k)/B_n|, \end{aligned}$$

$$\begin{aligned}
M &= \sup_n (1/B_n)|\sum_{k=1}^{n} w_k X_k| \\
&\le \sup_n (1/B_n)|\sum_{k=1}^{n} w_k d_k| + \sup_n (1/B_n)\sum_{k=1}^{n} w_k |Z_k - E_{k-1}(Z_k)| \\
&\le \sup_n |\sum_{k=1}^{n} (w_k/B_k) d_k| + \sup_n |\sum (w_k/B_k) d_k((B_n - B_k)/B_n)| \\
&+ \sup_n |\sum_{k=1}^{n} (w_k/B_k)|Z_k - E_{k-1}(Z_k)|| \\
&+ \sup_n |\sum_{k=1}^{n} (w_k/B_k)|Z_k - E_{k-1}(Z_k)|(B_n - B_k)/B_n| \\
&= I + II + III + IV.
\end{aligned}$$

By an inequality of B.Davis [2] there is a constant $B < \infty$ such that

$$\begin{aligned}
E(\sup_n |\sum_{k=1}^{n} (w_k/B_k) d_k|) &\le BE((\sum_{k=1}^{\infty} (w_k^2/B_k^2) d_k^2)^{1/2}) \\
&\le B(E(\sum_{k=1}^{\infty} (w_k^2/B_k^2) d_k^2))^{1/2} \\
&< \infty,
\end{aligned}$$

using Lemma 1 for the last step, and hence the expected values of I and II is finite. And

$$\begin{aligned}
&E(\sup_n \sum_{k=1}^{n} (w_k/B_k)|Z_k - E_{k-1}(Z_k)|) \\
&\le 2E \sum_{k=1}^{\infty} (w_k/B_k) E|Z_k| < \infty,
\end{aligned}$$

using Lemma 2 for the last step, and hence the expected values of III and IV is finite.

The following theorem is a generalization of Theorem 4 [3] and the proof is immediate from Theorem 3 using the same method as in Theorem 2.

Theorem 4. *If $E|X_1|log^+|X_1| < \infty$ and lim* sup *$N(x)/x < \infty$ as $n \to \infty$, then $M \in L^1$.*

Theorem 5. *Under the conditions of Theorem 1, we have $E(|S_n/B_n|) \to 0$ as $n \to \infty$.*

Proof. Write

$$X_n = d_n + (Z_n - E_{n-1}(Z_n))$$

as in the proof of Theorem 1. Then

$$\begin{aligned} E(|S_n/B_n|) &\leq E(|(1/B_n)\sum_{k=1}^{n} w_k d_k|) \\ &\quad +(1/B_n)\sum_{k=1}^{n} w_k E(|Z_k| + |E_{k-1}(Z_k)|) \\ &\leq ((1/B_n^2)\sum_{k=1}^{n} w_k^2 E(d_k^2))^{1/2} + (2/B_n)\sum_{k=1}^{n} w_k E|Z_k|. \end{aligned}$$

By Lemma 1 and 2, $\sum_{k=1}^{\infty}(w_k^2/B_k^2)Ed_k^2$ and $\sum_{k=1}^{\infty}(w_k/B_k)E|Z_k|$ are bounded. Thus by Kronecker's lemma, $E(|S_n/B_n|) \to 0$.

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