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FACTOR THEOREMS AND THEIR APPLICATION FOR N-GROUPS

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Dedicated to Professor Younki Chae on his 60th birthday

1. Introduction

Throughout this paper, a near-ring N is an algebraic system $(N, +, \cdot)$, where $(N, +)$ is a group, $(N, \cdot)$ is a semigroup and only one distributive law is postulated, we will consider the right distributive law. Sometimes, we will consider $N = N_0$ as zero-symmetric near-ring. An N-group M is a system ${}_NM$ where M is an additive group admitting scalar multiplication by the element of N with the properties: $(a+b)x = ax + bx$, $(ab)x = a(bx)$ for all a, b in N and all x in M. An N-module M is an N-group ${}_NM$ with the property that $a(x+y) = ax + ay$ for all a in N and all x, y in M. If $N = N_d$, then clearly N is an N-module. The other concepts of near-ring theory are known in G. Pilz [8].

If $f : M \to M'$ is an N-homomorphism, then Imf is an N-subgroup of M'. If Imf is an N-ideal of M' then f is call a normal N-homomorphism. Thus a normal N-homomorphism is an N-homomorphism.

2. Characterizations of Epic and Monic

We now state various characterizations of N-epimorphisms and monomorphisms analogous to those for surjections and injections in the category of sets and functions. For N-homomorphisms we have the advantage of the O-function, but we no longer can characterize, say, N-monomorphisms as we did injections by means of one-sided inverse.

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Proposition 2.1. *Let M and L be N-groups and let $f : M \to L$ be a normal N-homomorphism. then the following statements are equivalent:*

(1) *f is an N-epimorphism onto L (from time to time, epic);*

(2) *$Imf = L$;*

(3) *For every ${}_NK$ and every pair $g, h : L \to K$ of N-homomorphisms, $gf = hf$ implies $g = h$;*

(4) *For every ${}_NK$ and every N-homomorphism $g : L \to K$, $gf = 0$ implies $g = 0$.*

Proof. (1) $\Leftrightarrow$ (2) and (1) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). Let $h : L \to K$ be the zero N-homomorphism. Then $gf = 0$ means $gf = hf$; so assuming (3), we have $g = h = 0$.

(4) $\Rightarrow$ (2). Let $I = Imf$. Then $\pi : L \to L/I =$ Cokerf clearly satisfies $\pi f = 0$. So assuming (4) this means that $\pi = 0$. But since π is onto L/I, we have $L/I = 0$ whence $I = L$.

Proposition 2.2. *Let M and L be N-groups and $f : M \to L$ be an N-homomorphism. Then the following statements are equivalent:*

(1) *f is an N-monomorphism (from time to time, monic);*

(2) *$Kerf = 0$;*

(3) *For every ${}_NK$ and every pair $g, h : K \to M$ of N-homomorphisms, $fg = fh$ implies $g = h$;*

(4) *For every ${}_NK$ and every N-homomorphism $g : K \to M$, $fg = 0$ implies $g = 0$.*

Proof. The implication (4) $\Rightarrow$ (2) is the only one that offers any challenge. But let $K = Kerf$. Then $i : K \to M$ is an N-homomorphism and $f_i = 0$. So assuming (4) we have $i = 0$. But then $K = Imi = 0$.

Analogously, in ring and module theory we have the following result:

Remark 2.3. Let M and L be N-groups and let $f : M \to L$ be an N-homomorphism. Then f is an N-isomorphism if and only if there are functions $g, h : L \to M$ such that

$$fg = 1_L \text{ and } hf = 1_M.$$

When these last conditions are satisfied, $g = h$ is an N-isomorphism.

When $f : M \to L$ is an N-isomorphism, the unique N-homomorphism $g : L \to M$ satisfying the condition of (2.3) is inverse of f and is denoted by f^{-1}. Note that in (2.1) and (2.2) we did not claim as an equivalent condition the existence of one-sided inverses. As we shall see, this omission was not accidental.

3. The Factor Theorems

An N-homomorphism $f : M \to L$ that is the composite of N-homomorphisms

$$f = gh,$$

is said to factor through g and h. The following result essentially says that a homomorphism f factors uniquely through every epimorphism whose kernel is contained in that of f and through every monomorphism whose image contains the image of f.

Theorem 3.1 (The factor theorems). *Let M, M', L and L' be N-groups and let $f : M \to L$ be a normal N-homomorphism.*

(1) *If $g : M \to M'$ is an N-epimorphism with $Kerg \subset Kerf$, then there exists a unique homomorphism $h : M' \to L$ such that*

$$f = hg.$$

Moreover, $Kerh = g(Kerf)$ and $Imh = Imf$, so that h is monic iff $Kerg = Kerf$ and h is epic iff f is epic.

(2) *If $g : L' \to L$ is an N-monomorphism with $Imf \subset Img$, then there exists a unique homomorphism $h : M \to L'$ such that*

$$f = gh.$$

Moreover, $Kerh = Kerf$ and $Imh = g^{-1}(Imf)$, so that h is monic iff f is monic and h is epic iff $Img = Imf$.

$$\begin{array}{ccc} M & \xrightarrow{f} & L \\ {\scriptstyle g}\searrow & & \nearrow{\scriptstyle h} \\ & M' & \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{f} & L \\ {\scriptstyle h}\searrow & & \nearrow{\scriptstyle g} \\ & L' & \end{array}$$

(1) (2)

Proof. (1) Since $g : M \to M'$ is epic, for each $x' \in M'$ there is at least one $x \in M$ with $g(x) = x'$. If also $y \in M$ with $g(y) = x'$, then clearly

$x - y \in Kerg$. But since $Kerg \subset Kerf$, we have that $f(x) = f(y)$. Thus, there is a well defined function $h : M' \to L$ such that $f = hg$. To show that h is an N-homomorphism, let $x', y' \in M'$ and let $x, y \in M$ with $g(x) = x'$, $g(y) = y'$. Then for each $a, b \in N$,

$$g(ax + by) = ax' + by'$$

so that

$$h(ax' + by') = f(ax + by) = af(x) + bf(y) = ah(x') + bh(y').$$

The uniqueness of h with these properties is assured by (2.1.(3)), since g is an N-epimorphism. Moreover we show that $Kerh = g(Kerf)$: Let $h(x') = 0$ for $x' \in M'$. Then there exists x in M with $g(x) = x'$. Thus $hg(x) = f(x) = 0$, hence $x \in Kerf$, so that $x' = g(x) \in g(Kerf)$. Consequently $Kerh \subset g(Kerf)$. Conversely, let $g(x) \in g(Kerf)$. Then $f(x) = 0$ that is $hg(x) = 0$. It follows that $g(x) \in Kerh$. Thus $g(Kerf) \subset Kerh$. Next to prove that $Imh = Imf$; for all $x' \in M'$, $h(x') \in h(M')$ iff there exists x in M with $g(x) = x'$, $h(x') = hg(x) = f(x) \in f(M)$. The final assertion is trivial.

(2) For each $x \in M$, $f(x) \in Imf \subset Img$. So since g is monic, there is a unique $y' \in L'$ such that $g(y') = f(x)$. Therefore we can define a function $h : M \to L'$ by $h(x) = y'$, where $g(y') = f(x)$. Then h is a well defined function. Indeed, if $x_1 = x_2$ by $h(x_1) = y_1'$, $h(x_2) = y_2'$, then $f(x_1) = f(x_2) \in Imf \subset Img$, so that $g(y_1') = g(y_2')$. Since g is monic $y_1' = y_2'$ that is $h(x_1) = h(x_2)$. The rest of the proof is also easy.

Let M be an N-group and $X \subset M$ and $A \subset N$. Then any element M of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = \sum_{i=1}^{n} a_ix_i$$

with $x_1, \cdots, x_n$ in M and $a_1, \cdots, a_n$ in A is called a linear combination of X with coefficients in A, or simply an A-linear combination of X. We shall denote the set of all such A-linear combination of X by AX.

Lemma 3.2. *Let M be an N-space (N-module) and let X be a non-empty subset of M. Then NX is an N-subspace (N-ideal) of M.*

Proof. The N-linear combinations of X are closed under the group operation of M, and the identity

$$a(r_1x_1 + \cdots + r_nx_n) = (ar_1)x_1 + \cdots + (ar_n)x_n$$

for any $a \in N$, $r_1x_1 + \cdots + r_nx_n \in NX$, finishes the job.

To avoid a special case later, we agree

$$N\phi = 0,$$

that is, 0 is the unique N-linear combination of ϕ. The following, which is an easy exercise, characterizes submodules as those non-empty subsets "closed" under all N-linear combinations.

Lemma 3.3. *Let M be a unital N-space (N-module) and let L be a non-empty subset of M. Then the following are equivalent:*

(1) *L is an N-subspace(N-ideal) of M;*

(2) *$NL = L$;*

(3) *For all $a, b \in N$ and all $x, y \in L, ax + by \in L$.*

Let $N = N_0$. M an N-group. Then the set $\rho(M)$ of all ideals of M is a complete modular lattice with respect to $\subset$. In this lattice, if $\mathcal{A}$ is a non-empty subset of $\rho(M)$, then its join and meet are given by

$$\sum \mathcal{A} \text{ and } \cap \mathcal{A},$$

respectively. In particular, if K and L are ideals of M, then

$$K + L \text{ and } K \cap L$$

are their join and meet, respectively; if H is another ideal of M, then

$$K \subset H \text{ implies } H \cap (K + L) = K + (H \cap L).$$

Given an N-group M and a subset $X \subset M$, the set $\mathcal{A}$ of all ideals of M that contain X contains M and so is non-empty. Its intersection $\cap\mathcal{A}$ is again an ideal of M and it is in fact, the unique smallest N-ideal of M that contain X. we call it the N-ideal of M spanned by X.

Lemma 3.4. *If M is a unitial N-space and if X is a subset of M, then the subspace of M spanned by X is just NX, the set of all N-linear combinations of X.*

Proof. By (3.2) NX is a subspace of M and since $1x = x$ for all $x \in M$, we certainly have $X \subset NX$. Finally, by (3.3), any subspace that contains X must contain the linear combinations NX.

If $(M_i)_{i\in I}$ are N-ideals of M, then $\sum_{i\in I} M_i$ is the ideal spanned by $(M_i)_{i\in I}$. Thus if

$$M = \sum_{i\in I} M_i,$$

then we say that the ideals $(M_i)_{i\in I}$ span M. If X is a subset of ${}_NM$ such that

$$NX = M,$$

then X is said to span M, and X is called a spanning set for M. An N-module with a finite spanning set is said to be finitely spanned (or finitely generated). A unitary N-group with a single element spanning set is a cyclic N-group (or principal N-group). Thus a cyclic N-group is one of the form $M = N\{x\} \equiv Nx$ where x is some element of M; and we write

$$M = Nx = \{nx | n \in N\}$$

of course, any near-ring with identity is cyclic. Now it is clear that every N-module is spanned by the set of its cyclic N-ideals: that is, if X is a spanning set for ${}_NM$, then

$$M = \sum_{x\in X} N_x.$$

An N-group M is called simple in case M has no non-trivial ideals and is called irreducible in case it has no N-subgroups except N_0 and M.

Remark 3.5.. Let N be a zero-symmetric near-ring. Then every irreducible N-group ${}_NM$ is always simple.

Proof. Let L be an N-ideal of M. Then for any $x \in L$ and any $a \in N$, we have that $ax = a(0 + x) - a0 \in L$, that is, L is an N-subgroup of M. Consequently irreduciblity implies simplicity for every N-subgroup.

Let K be an N-ideal of an N-group M. Then it is easy to see that the set

$$\rho(M)/K = \{H \in \rho(M) | K \leq H\}$$

is a sublattice of $\rho(M)$. Moreover, for each H in this sublattice

$$\pi(H) = H/K$$

is obviously an N-ideal of the factor N-group M/K. Since clearly $H \leq H'$ implies $\pi(H) \leq \pi(H')$, we have that π defines an ordering preserving function from $\rho(M)/K$ into $\rho(M/K)$. On the other hand, if T is an N-ideal of M/K, then

$$\pi^{-1}(T) = \{x \in M | x + K \in T\}$$

is an N-ideal of M, and since $o + K = k + K \in T$ for all $k \in K$, clearly $K \leq \pi^{-1}(T)$. We see at once that $\pi\pi^{-1}(T) = T$ and $\pi^{-1}\pi(H) \geq H$ for all $T \in \rho(M/K)$ and for all $H \in \rho(M)/K$. But if $x \in \pi^{-1}\pi(H)$, then $x + K = a + K$ for some $a \in H$ and so since $K \leq H$, we have $x \in H$. Thus π and π^{-1} define inverse bijections. Finally, since π^{-1} is also order-preserving, we have the following statement:

Theorem 3.6. *Let M be an N-group and let K be an N-ideal of M. Then the lattice of ideals of the factor N-group M/K is lattice isomorphic to the lattice of ideals of M that contain K via the inverse maps*

$$\pi : H| \sim\sim\rightarrow H/K = \{x + K | x \in H\}$$

$$\pi^{-1} : T| \sim\sim\rightarrow \pi^{-1}(T) = \{x \in M | x + K \in T\}.$$

Since an N-group is simple iff its lattice of ideals is a two element chain, so we have the following fact.

Corollary 3.7. *An N-factor group M/K is simple if and only if K is a maximal ideal of M.*

As applications of the first part of the factor theorems we have the very important isomorphism theorems.

Corollary 3.8. *Let M and L be N-groups. Then we have the following facts:*

(1) *If $f : X \rightarrow L$ is an epimorphism with $Kerf = K$, then there is a unique isomorphism $\phi : M/K \rightarrow L$ such that $\phi(x + K) = f(x)$ for all $x \in M$.*

(2) *If $K \leq L \leq M$, then $M/L \cong (M/K)(L/K)$.*

(3) *If $H \leq M$ and $K \leq M$, then $(H + K)/K \cong H/(H \cap K)$.*

Proof. (1) Let $f : M \rightarrow L$ be an epic and $Kerf = K$. Put $M' = M/K$ and g is the natural epic $g = \pi : M \rightarrow M/K$. By the factor theorem (3.1.1), we have the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{f} & L \\ {}^{g}\searrow & & \nearrow^{\phi} \\ & M/K & \end{array}$$

so there exists a unique N-homomorphism $\phi : M/K \to L$ such that $\phi g = f$. Moreover since $Kerf = Kerg$, ϕ is monic and since f is epic, ϕ is epic. Thus we are done.

To prove (2) and (3) apply (1) to the epimorphism $f' : M/K \to M/L$ via $f'(x+K) = x + L$ and to the epimorphism $f'' : H \to (H+K)/K$ via $f''(h) = h + K$ respectively.

Corollary 3.9. *Let M and N be N-groups and let $f : M \to L$ be an N-epimorphism with Kernel K. Then*

$$H| \rightsquigarrow f(H) = \{f(x) | x \in H\}$$

$$P| \rightsquigarrow f^{-1}(P) = \{x \in M | f(x) \in P\}$$

are inverse lattice isomorphisms between the lattice $\rho(M)/K$ and the lattice $\rho(L)$ of all ideals of L.

Proof. By the (3.8.1) we have an isomorphism $\phi : M/K \to L$ such that

$$\begin{array}{ccc} M & \xrightarrow{f} & L \\ {}^{\pi}\searrow & & \nearrow^{\phi} \\ & M/K & \end{array}$$

commutes. Clearly, ϕ induces a lattice isomorphism between $\rho(M/K)$ and $\rho(L)$. But by theorem (3.6) π induces one between $\rho(M)/K$ and $\rho(M/K)$.

A minimal N-ideal of a near-ring N is an N-ideal which is minimal in the set of all non-zero N-ideals.

Theorem 3.10. *Let N be a near-ring with identity such that N is a finite direct sum of minimal N-ideals. Then every N-ideal A in N is of the form eN where e is an idempotent in N.*

Proof. By hypothesis $N = M_1 \oplus M_2 \oplus \cdots \oplus M_k$ where M_i are minimal ideals in N and assume that $A \neq 0$ and $A \neq N$. Since $A \cap M_i \subset M_i$ and M_i are also minimal ideals, either $A \cap M_i = 0$ or $A \cap M_i = M_i$. Since $A \neq N$, there exists, say, M_i (after renumbering if necessary) such that

$M_1 \subset A$. But then $M_1 \cap A = 0$. So $A_1 = A + M_1$ is a direct sum. If $A_1 \neq N$, then there exists some M_1, say M_2 such that

$$A_1 \cap M_2 = 0 \text{ and } A \subset A_2 = A_1 \oplus M_2 = A \oplus M_1 \oplus M_2.$$

By continuing this process, we must come to an ideal, say A_s, which contains all M_i and therefore coincides with N. Thus there exists an ideal B such that

$$N = A \oplus B.$$

Since $1 \in N$, write $1 = e + f$ where $e \in A$ and $f \in B$. It follows that

$$e = e^2 + fe \text{ that is } -e^2 + e = fe \in A \cap B = 0.$$

Thus

$$e = e^2 \text{ and } fe = 0. \text{ Similarly } f = f^2, ef = 0.$$

If $a \in A$ then $a = ea + fa$. This gives $-ea + a = fa \in A \cap B = 0$. Hence $a = ea$ and so $A \subset eN$ and since $e \in A$, $eN \subset A$. This shows that $A = eN$ as desired. Moreover $B = fN = (-e+1)N$ is easily obtained.

Note that the corresponding statement of theorem 3.10 hold for "right ideals" instead of "ideals". As a final application here of the factor theorems, we give

Corollary 3.11. *Let M and K be N-groups and let $j : K \to M$ be an N-monomorphism with $Imj = I$. Then there is a unique isomorphism $\psi : I \to K$ such that $j\psi = i$ where $i : I \to M$ is the inclusion map.*

Proof. Let $I = M$, $M = L$, $K = L'$ and $i = f$, and $j = g$ in (3.1.2)

Remark 3.12. An exact sequence of the form

$$0 \longrightarrow K \xrightarrow{f} M \xrightarrow{g} L \longrightarrow 0$$

is called a short exact sequence. In this sequence f is monic and g is epic. Thus by (3.8.1) and (3.11) there exist unique isomorphisms ψ and ϕ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{f} & M & & \xrightarrow{g} & L & \longrightarrow & 0 \\ & & \psi\nwarrow & & \nearrow \ \pi\searrow & & & \nearrow\phi & & \\ & & & Imf & & M/Kerf & & & & \end{array}$$

commutes where i is the inclusion map and π is the natural epimorphism. But by exactness $Imf = Kerg$, so ψ and ϕ are N-isomorphisms such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \stackrel{f}{\longrightarrow} & M & \stackrel{g}{\longrightarrow} & L & \longrightarrow & 0 \\ & & \psi \downarrow & & 1_M \downarrow & & \pi \downarrow & & \\ 0 & \longrightarrow & Imf & \stackrel{i}{\longrightarrow} & M & \stackrel{\pi}{\longrightarrow} & M/Imf & \longrightarrow & 0 \end{array}$$

commutes. That is, every short exact sequence is "isomorphic" in this latter sense to one of the form

$$0 \longrightarrow M' \stackrel{i}{\longrightarrow} M \stackrel{\pi}{\longrightarrow} M/M' \longrightarrow 0$$

where i is an inclusion map of an N-ideal M' of M and π is the natural N-epimorphism.

References

[1] Anderson, F.W. and Fuler, K.R. *Rings and Categories of Modulers*, Springer-Verlag, New York, Heidel berg, Berlin, (1974).

[2] Bhattacharya, P.B. and Jain, S.K. *First Course in Rings, Fields and Vector Spaces*, Wiley Eastern Limited, (1977).

[3] Cho, Y.U. *Properties of Exactness and Projectivity of N-modules*, J. of Pusan Women's Univ, vol.31, (1991), 107-120.

[4] Hu, Sze Tsen, *Homological Algebra*, Holden-day Inc. San Francisco, Cambridge, London, Amsterdam, (1968).

[5] Lambek, J., *Lectures on Rings and Modules*, Chelsea Publishing Company, New York, (1976).

[6] Maxson, C.J., *Dickson Near-Rings*, Journal of Algebra vol.14, (1970), 152-169.

[7] Oswald, A., *Semisimple Near-Rings have the Maximum Condition on N-subgroups*, J. of London Math. Soc (2), Vol.11(1975), 408-412.

[8] Pilz, G. *Near-Rings*, North-Holland Publishing Company Amsterdam, New York, Oxford, (1983).

[9] Roth, R.J., *The Structure of Near-Rings and Near-Ring Modules*, Doctoral Dissertation, Duke Univ. (1962).

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