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EXPONENTIAL BEHAVIOR OF SOLUTIONS FOR PERTURBED LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Dedicated to Dr. Y.K. Chae for his 60th birthday

1. Introduction

The qualitative properties for linear differential equations with perturbations have been intensively investigated. Among qualitative theorems, Lyapunov's stability theorems and Perron's theorems are most popular in the literatures[1,2,3,5].

In this paper, we are concerned with generalizations of Perron type stability theorems. That is, the main concern is that under what conditions for the linear parts and/or the perturbation parts of the differential equations, the solutions are stable or asymptotically stable in a suitable sense. In [11], T. Taniguchi gives a partial answer for this question by generalizing a Perron's theorem as follows:

Theorem[11]. *Assume that the following conditions hold;*

(a) $\|f(t,x)\| \leq F(t,\|x\|), F(t,0) \equiv 0,$ *and* $F(t,u)$ *is monotone nondecreasing with respect to* u *for each fixed* $t \geq 0$,

(b) $F(t,u) \in C[[0,\infty) \times B_\delta^+, R^+]$.

(c) *the zero solution of the linear differential equation ;* $dx/dt = A(t)x$ *is uniformly stable, that is, there exists a constant* $K \geq 1$ *such that* $\|U(t)U^{-1}(s)\| \leq K, t \geq s \geq 0$, *where* $U(t)$ *is the fundamental matrix solution of* $dx/dt = A(t)x$.

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If the zero solution of the differential equation ; $dt/dt = KF(t,y)$ is uniformly stable (uniform-asymptotically stable), then the zero solution of the perturbed equation $dx/dt = A(t)x + f(t,x)$ is also uniformly stable (uniform-asymptotically stable).

The main purpose of this paper is to generalize in a sense the above Theorem by the T(m)-stability concepts and also to present some T(m)-boundedness theorems.

In section 2 we discuss T(m)-stability of the zero solution of the perturbed equation, and in section 3 we present some T(m)-boundedness theorems.

2. T(m)-stability theorems

Let R^n and R^+ be the n-dimensional Euclidean space and the set of all non-negative real numbers, respectively. $C[X,Y]$ denotes the set of all continuous mappings from a topological space X into a topological space Y. Let $A(t)$ be a continuous $n \times n$ martrix - valued function defined on $[0.\infty)$ and let $f(t,x) \in C[[0,\infty) \times R^n, R^n]$.

Consider a linear differential equation:

$$dx/dt = A(t)x \tag{2.1}$$

and a pertubed differential equation of (2.1):

$$ds/dt = A(t)x + f(t,x). \tag{2.2}$$

Let $U(t)$ be the fundamental matrix solution of (2.1). Then the solution $x(t)$ of (2.2) satisfies the integral equaion:

$$x(t) = U(t)U^{-1}(t_0)x_0 + \int_{t_0}^{t} U(t)U^{-1}(s)f(s,x(s))ds, t \geq t_0.$$

Let us introduce some stability concepts. Let $g \in C[[0,\infty) \times R^n, R^n]$ be given. Consider a differential equation of a general form:

$$dy/dt = g(t,y). \tag{2.3}$$

Let $y(t) \equiv y(t : t_0, y_0)$ denote a solution of (2.3) with an initial value (t_0, y_0). Assume $g(t,0) = 0$ for all $t \geq t_0$.

We give T(m)-stability definitions of solutions of (2.3).

Definition 1. Let m be a real number. The zero solution of (2.3) is said to be

(i) $(T(m)-S)$ *T(m)-stable if for any* $\epsilon > 0$ *and any* $t_0 \geq 0$, *there exists a* $\delta(t_0,\epsilon) > 0$ *such that if* $\|y(t_0)\| < \delta(t_0,\epsilon)$, *then* $\|y(t)e^{-mt}\| < \epsilon$ *for all* $t \geq t_0$,

(ii) $(T(m)-US)$ *T(m)-uniformly stable* if the $\delta(t_0,\epsilon)$ in $(T(m)-S)$ is independent of time t_0,

(iii) $(T(m)-QAS)$ *T(m)-quasi-asymptotically stable* if for any $\epsilon > 0$ and any $t_0 \geq 0$, there exist a $T(t_0,\epsilon) > 0$ and a $\delta(t_0,\epsilon) > 0$ such that if $\|y(t_0)\| < \delta(t_0,\epsilon)$ then $\|y(t)e^{-m(t-t_0)}\| < \epsilon$ for all $t \geq t_0 + T(t_0,\epsilon)$,

(iv) $(T(m)-QUAS)$ *T(m)-quasi-uniform-asymptotically stable* if the $T(t_0,\epsilon)$ and the $\delta(t_o,\epsilon)$ in $(T(m)-QAS)$ are independent of time t_0,

(v) $T(m)-AS)$ *T(m)-asymptotically stable* if it is T(m)-stable and is T(m)-quasi-asymptotically stable,

(vi) $(T(m)-UAS)$ *T(m)-uniform-asymptotically stable* if it is T(m)-uniformly stable and is T(m)-quasi-uniform-asymptotically stable.

Remark. The T(m)-stability concepts are exactly the same as the usual definitions of stability when $m = 0$. Now we present a lemma for integral inequalities which plays a key role for our theorems.

Lemma 1[9 ,p.315]. *Let the following condition (i) or (ii) hold for functions* $f(t), g(t) \in C[[t_0,\infty)\times R^+, R^+]$:

$$(i) \qquad f(t) - \int_{t_0}^{t} F(s,f(s))ds \leq g(t) - \int_{t_0}^{t} F(s,g(s))ds, t \geq t_0$$

and $F(s,u)$ *is strictly increasing in* u *for each fixed* $s \geq 0$,

$$(ii) \qquad f(t) - \int_{t_0}^{t} F(s,f(s))ds < g(t) - \int_{t_0}^{t} F(s,g(s))ds, t \geq t_0$$

and $F(s,u)$ *is monotone nondecreasing in* u *for each fixed* $s \geq 0$.

If $f(t_0) < g(t_0)$, *then* $f(t) < g(t), t \geq t_0$.

Throughout this section we assume that $f(t,0) \equiv 0$.

Thus the equation (2.2) has the zero solution. Now we have a main result:

Theorem 1. *Let the following conditions hold for the differential equation* (2.2):

(1a) $\|f(t,x)\| \leq F(t,\|x\|), F(t.0) \equiv 0$ *and* $F(t,u)$ *is monotone nondecreasing with respect to* u *for each fixed* $t \geq 0$, (1b) $F(t,u) \in C[[0,\infty) \times R^+, R^+]$,

(1c) *the zero solution of the differential system* (2.1) *is* $T(m_1)$ *-uniformly stable for an* $m_1 \leq 0$. *that is, there exists a constant* $K \geq 1$ *such that* $\|U(t)U^{-1}(s)\| \leq Ke^{m_1 t}, t \geq s \geq 0$.

If the zero solution of the differential equation

$$dy/dt = KF(t,y) \tag{2.4}$$

is $T(m_2)$*-stable for a real number* m_2*, then the zero solution of* (2.2) *is* $T(m_1+m_2)$*-stable.*

Proof. Let $x(t) \equiv x(t : t_0, x_0)$ be a solution of (2.2) with an initial value $(t_0, x_0), t_0 \geq 0$. Then the solution $x(t)$ is of the form

$$x(t) = U(t)U^{-1}(t_0)x_0 + \int_{t_0}^{t} U(t)U^{-1}(s)f(s,x(s))ds. \tag{2.5}$$

Thus we obtain that from condition (1c)

$$\|x(t)\| \leq Ke^{m_1 t}\|x_0\| + \int_{t_0}^{t} Ke^{m_1 t}\|f(s,x(s))\|ds,$$

and from condition (1a)

$$\begin{aligned} \|x(t)\|e^{-m_1 t} &\leq K\|x_0\| + \int_{t_0}^{t} K\|f(s,x(s))\|ds \\ &\leq K\|x_0\| + \int_{t_0}^{t} KF(s,\|x(s)\|)ds \\ &\leq K\|x_0\| + \int_{t_0}^{t} KF(s,\|x(s)\|e^{-m_1 s})ds \end{aligned}$$

Next let $y(t) \equiv y(t : t_0, y_0)$ be the solution of (2.4) passing through (t_0, y_0) and let $K\|x_0\|e^{-m_1 t_0} < y_0$. Then we have

$$\|x(t)\|e^{-m_1 t} - \int_{t_0}^{t} KF(s,\|x(s)\|e^{-m_1 s})ds < y(t) - \int_{t_0}^{t} KF(s,y(s))ds.$$

Therefore applying Lemma 1, we obtain

$$\|x(t)\|e^{-m_1 t} < y(t), t \geq t_0.$$

Since the zero solution of (2.4) is $T(m_2)$-stable, for any $\epsilon > 0$ there exists a $\delta_0(t_0,\epsilon) > 0$ such that if $|y_0| < \delta_0(t_0,\epsilon)$, then $|y(t)e^{-m_2 t}| < \epsilon$ for all

$t \geq t_0$. Thus set $\delta(t_0, \epsilon) = \delta_0(t_0, \epsilon)/Ke^{-m_1 t_0}$. If $\|x_0\| < \delta(t_0, \epsilon)$, then take a $y_0 > 0$ such that $K\|x_0\|e^{-m_1 t_0} < y_0 < \delta_0(t_O, \epsilon)$.

Hence we have

$$\|x(t)e^{-(m_1+m_2)t}\| = \|x(t)e^{-m_1 t}\|e^{-m_2 t}$$
$$< y(t)e^{-m_2 t} < \epsilon$$

for all $t \geq t_0$. This completes the proof.

Corollary 1. *Let conditions (1a),(1b), and (1c) hold for the differential equation* (2.2). *If the zero solution of the differential equation* (2.4) *is $T(m_2)$-stable for a real number m_2 satisfying $m_1 + m_2 \leq 0$, then the zero solution of* (2.2) *is stable.*

Corollary 2. *Let conditions (1a), (1b) and the following condition hold for the differential equation (2.2):*

(1d) *the zero solution of the differential equation (2.1) is uniformly stable, that is, there exists a constant $K \geq 1$ such that $\|U(t)U^{-1}(s)\| \leq K, t \geq s \geq 0$.*

If the zero solution of the differential equation (2.4) is $T(m)$-uniformly stable for a real number m, then the zero solution of (2.2) is also $T(m)$-uniformly stable.

Furthermore, if the zero solution of (2.4) is $T(m)$-uniformly stable for a real number $m \leq 0$, then the zero solution of (2.2) is uniformly stable.

Example 1. Let the following conditions hold for the differential equation (2.2):

(1e) $\|f(t,x)\| \leq a(t)\|x\|$,

(1f) $a(s) \in C[[0,\infty), R^+]$ and there exists a positive constant M such that

$$\sup_{t\geq 0}\frac{\int_0^t a(s)ds}{t} \leq M.$$

If the zero solution of the differential equation (2.1) is uniformly stable, then the zero solution of (2.2) is T(m)-uniformly stable for a real number $m \geq KM$.

Proof. Set $F(t,u) \equiv a(t)u,\ u \geq 0$. First of all, we show that

$$dy/dt = Ka(t)y \tag{2.4}$$

is T(m)-uniformly stable for an $m \geq KM$.

Let $y(t) \equiv y(t : t_0, y_0), t \geq t_0 \geq 0$ be a solution of (2.6) passing through (t_0, y_0). Then we obtain

$$y(t) = y_0 e^{\int_{t_0}^{t} Ka(s)ds} \leq y_0 e^{KMt}.$$

Set $\delta(\epsilon) \equiv \epsilon/2$. If $|y_0| < \delta(\epsilon)$, then $|y(t)e^{-mt}| \leq |y_0 e^{(KM-m)t}| < \epsilon$ for all $t \geq 0$, which implies that the zero solution of (2.6) is T(m)-uniformly stable for $m \geq KM$. Therefore, since conditions (1a), (1b) and (1d) of Corollary 2 are satisfied, the zero solution of (2.2) is T(m)-uniformly stable for a real number $m \geq KM$.

Next we discuss T(m)-asymptotic stability properties.

Lemma 2. *Let conditions (1a), (1b) and (1c) hold for the differential equation(2.2). If the zero solution of the differential equation (2.4) is $T(m_2)$-quasi-asymptotically stable for a real number m_2, then the zero solution of (2.2) is $T(m_1 + m_2)$-quasi-asymptotically stable.*

Proof. From the proof of Theorem 1, we have

$$\|x(t)\|e^{-m_1 t} < y(t), t \geq t_0.$$

Let any $\epsilon > 0$ and any $t_0 \geq 0$ be given. By the assumption, there exist a $\delta_0(t_0, \epsilon) > 0$ and a $T(t_0, \epsilon) > 0$ such that if $|y_0| < \delta_0(t_0, \epsilon)$, then $|y(t)e^{-m_2(t-t_0)}| < \epsilon, \quad t \geq t_0 + T(t_0, \epsilon)$.

Set $\delta(t_0, \epsilon) \equiv \delta_0(t_0, \epsilon)/Ke^{-m_1 t_0}$. If $\|x_0\| < \delta(t_0, \epsilon)$, then take a $y_0 > 0$ such that $K\|x_0\|e^{-m_1 t_0} < y_0 < \delta_0(t_0, \epsilon)$. Therefore we obtain

$$\|x(t)e^{-(m_1+m_2)(t-t_0)}\| = \|x(t)e^{-m_1 t}\|e^{-m_2(t-t_0)}e^{m_1 t_0}$$
$$< y(t)e^{-m_2(t-t_0)}e^{m_1 t_0} < \epsilon$$

for all $t \geq t_0 + T(t_0, \epsilon)$. This completes the proof.

Theorem 2. *Let conditions (1a),(1b) and (1c) hold for the differential equation (2.2). If the zero solution of the differential equation (2.4) is $T(m_2)-$asymptotically stable for a real number m_2, then the zero solution of (2.2) is $T(m_1 + m_2)$-asymptotically stable.*

Proof. By Theorem 1 and Lemma 2, the zero solution of (2.2) is $T(m_1 + m_2)$-stable and is $T(m_1 + m_2)$-quasi-asymptotically stable, respectively. Thus by Definition 1, the zero solution of (2.2) is $T(m_1+m_2)$ -asymptotically stable.

Corollary 3. *Let conditions (1a),(1b) and (1c) hold for the differential equation (2.2). If the zero solution of the differential equation (2.4) is $T(m_2)$-asymptotically stable for a real number m_2 satisfying $m_1 + m_2 \leq 0$, then the zero solution of (2.2) is asymptotically stable.*

Corollary 4. *Let conditions (1a), (1b) and (1d) hold for the differential equation (2.2). If the zero solution of the differential equation (2.4) is $T(m)$-uniform-asymptotically stable for a real number m, then the zero solution of (2.2) is also $T(m)$-uniform-asymptotically stable.*

Furthermore, if the zero solution of (2.4) is $T(m)$-uniform-asymptotically stable for an $m \leq 0$, then the zero solution of (2.2) is uniformly asymptotically stable.

3. T(m)-boundedness theorems

In this section we present boundedness theorems of the differential equation (2.2).

Now we give T(m)-boundedness definitions of solutions of the differential system (2.3).

Definition 2. Let m be a real number. The solutions of (2.3) are said to be

(i) (T(m)-EB) *T(m)-equibounded* if for any $\rho > 0$ and any $t_0 \geq 0$, there exists a $\beta(t_0, \rho) > 0$ such that if $\|y_0\| < \rho$, then $\|y(t)e^{-mt}\| < \beta(t_0, \rho)$ for all $t \geq t_0$.

(ii) (T(m)-UB) *T(m)-uniformly bounded* if the $\beta(t_0, \rho)$ in (T(m)-EB) is independent of time t_0.

(iii) (T(m)-EUB) *T(m)-equiultimately bounded* if there exists a $\beta > 0$, and for any $\rho > 0$ and any $t_0 \geq 0$, there exists a $T(t_0, \rho) > 0$ such that if $\|y_0\| < \rho$, then $\|y(t)e^{-m(t-t_0)}\| < \beta$ for all $t \geq t_0 + T(t_0, \rho)$,

(iv) (T(m)-UUB) T(m)-uniform-ultimately bounded if the $T(t_0, \rho)$ in (T(m)-EUB) is independent of time t_0.

Remark. The T(m)-boundedness concepts are the usual definitions of boundedness when $m = 0$. We prove boundedness properties of the differential equation (2.2).

Theorem 3. *Let conditions (1a),(1b) and (1c) hold except that $F(t, 0) \equiv 0$ for the differential equation (2.2).*

If the solutions of the differential equation (2.4) are $T(m_2)$-equibounded for a real number m_2, then the solutions of (2.2) are $T(m_1+m_2)$-equibounded.

Proof. Let $x(t) \equiv x(t : t_0, x_0)$ be the solution of (2.2) and let $y(t) \equiv y(t : t_0, t_0)$ be the solution of (2.4). Let $K\|x_0\|e^{-m_1 t_0} < y_0$. Then we obtain that in the same way as in the proof of Theorem 1

$$\|x(t)e^{-m_1 t}\| < y(t), t \geq t_0.$$

Since the solutions of (2.4) are $T(m_2)$-equibounded, for any $\rho > 0$ and any $t_0 \geq 0$, there exists a $\beta_0(t_0, \rho) > 0$ such that if $|y_0| < \rho$, then $|y(t)e^{-m_2 t}| < \beta_0(t_0, \rho)$ for all $t \geq t_0$. Thus set $\beta(t_0, \rho) \equiv \beta_0(t_0, Ke^{-m_1 t_0}\rho$. If $\|x_0\| < \rho$, then take a $y_0 > 0$ such that $Ke^{-m_1 t_0}\|x_0\| < y_0 < Ke^{-m_1 t_0}\rho$. Hence it follows that

$$\begin{aligned} \|x(t)e^{-(m_1+m_2)t}\| &= \|x(t)e^{-m_1 t}\|e^{-m_2 t} \\ &< y(t)e^{-m_2 t} \\ &< \beta(t_0, \rho) \end{aligned}$$

for all $t \geq t_0$. This completes the proof.

Corollary 5. *Let conditions (1a),(1b) and (1c) hold except that $F(t, 0) \equiv 0$ for the differential equation (2.2).*

If the solutions of the differential equation (2.4) are $T(m_2)$-equibounded for m_2 satisfying $m_1 + m_2 \leq 0$, then the solutions of (2.2) are equibounded.

Corollary 6. *Let conditions (1a), (1b) and (1d) hold except that $F(t, 0) \equiv 0$ for the differential equation (2.2).*

If the solutions of the differential equation (2.4) are $T(m)$-uniformly bounded for a real number m, then the solutions of (2.2) are also $T(m)$-uniformly bounded.

Furthermore, if the sloutions of (2.4) are $T(m)$-uniformly bounded for $m \leq 0$, then the solutions of (2.2) are uniformly bounded.

Theorem 4. *Let (1a),(1b) and (1c) hold except that $F(t, 0) \equiv 0$ for (2.2). If the solutions of the differential equation (2.4) are $T(m_2)$-equiultimately bounded for a real number m_2, then the solutions of (2.2) are $T(m_1 + m_2)$-equiultimately bounded.*

Proof. From the proof of theorem 1, we have

$$\|x(t)\|e^{-m_1 t} < y(t), t \geq t_0.$$

Since the solutions of (2.4) are $T(m_2)$ equiultimately bounded, there exist a $\beta > 0$, and for any $\rho_0 > 0$ and any $t_0 \geq 0$ there exists a $T_0(t_0, \rho) > 0$

such that if $|y_0| < \rho_0$, then $|y(t)e^{-m_2(t-t_0)}| < \beta$ for all $t \geq t_0 + T_0(t_0, \rho)$. Thus set $T(t_0, \rho) = T_0(t_0, e^{-m_1 t_0}\rho)$. If $\|x_0\| < \rho$, then take a $y_0 > 0$ such that $Ke^{-m_1 t_0}\|x_0\| < y_0 < Ke^{-m_1 t_0}\rho$. Therefore it follows that

$$\|x(t)e^{-(m_1+m_2)(t-t_0)}\| = \|x(t)e^{-m_1 t}\|e^{-m_2(t-t_0)}e^{m_1 t_0}$$

$$\leq y(t)e^{-m_2(t-t_0)} < \beta$$

for all $t \geq t_0 + T(t_0, \rho)$. This completes the proof.

Corollary 7. *Let conditions (1a), (1b) and (1c) hold except that* $F(t, 0) \equiv 0$ *for the differential equation(2.2).*

If the solutions of the differential equation (2.4) are $T(m_2)$*-equiultimately bounded for* m_2 *satisfying* $m_1 + m_2 \leq 0$*, then the solutions of (2.2) are equiultimately bounded.*

Corollary 8. *Let conditions (1a), (1b) and (1d) hold except that* $F(t, 0) \equiv 0$ *for the differential equation (2.2). If the solutions of the differential equation (2.4) are* $T(m)$*-uniform-ultimately bounded for a real number* m*, then the salutions of (2.2) are also* $T(m)$*-uniform-ultimately bounded.*

Furthermore, if the solutions of (2.4) are $T(m)$*-uniform-ultimately bounded for* $m \leq 0$*, then the solutions of (2.2) are uniform-ultimately bounded.*

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