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MANIFOLDS WITH KAEHLER-BOCHNER METRIC

U-Hang Ki and Byung Hak Kim

Dedicated to Professor Younki Chae on his 60th birthday

It is known that if a manifold with Kaehler-Bochner metric has constant scalar curvature, then M is either a space of constant holomorphic sectional curvature or a locally product space of two spaces of constant holomorphic sectional curvature c and $-c(\geq 0)$. This work is to prove that the scalar curvature is constant if and only if the trace of S^m is constant in a manifold with parallel Bochner curvature tensor, where S is the Ricci operator.

This result is applied to the manifolds with Kaehler-Bochner metric and we get generalized theorems of known facts.

1. Introduction

Let M be a Kaehlerian manifold of real dimension n with almost complex structure J and Kaehler metric g.

Bochner [1] introduced the so called Bochner curvature tensor B on M defined by

$$\begin{aligned} B(X,Y) &= R(X,Y) - \frac{1}{n+4}\{SX\wedge Y + X\wedge SY + SJX\wedge JY \\ &\quad +JX\wedge SJY - 2g(JX,SY)J - 2g(JX,Y)SJ\} \\ &\quad +\frac{r}{(n+2)(n+4)}\{X\wedge Y + JX\wedge JY - 2g(JX,Y)J\} \end{aligned} \tag{1.1}$$

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for any vector fields X and Y, where R, S and r are the Riemannian curvature tensor, the Ricci operator which is a symmetric (1,1) tensor and the scalar curvature of M respectively. $X \wedge Y$ denotes the endomorphism

$$X \wedge Y : Z \to g(Y,Z)X - g(X,Z)Y.$$

We say that g is a Kaehler-Bochner metric if B vanishes on the Kaehlerian manifold. Matsumoto and Tanno [3] proved

Theorem A. *Let M be a manifold with Kaehler-Bochner metric. If the scalar curvature of M is constant, then M is either a space of constant holomorphic sectional curvature or a locally product space of two spaces of constant holomorphic sectional curvature c and $-c(\geq 0)$.*

The purpose of this paper is to prove that the scalar curvature is constant if and only if the trace of S^m is constant for an integer $m > 1$ in a manifold with parallel Bochner curvature tensor, henceforth it holds in a manifold with Kaehler-Bochner metric. By use of this result, we have the following generalization of Theorem A.

Theorem. *Let M be a manifold with Kaehler-Bochner metric and the trace of S^m is constant for a positive integer m. Then M is a space of constant holomorphic sectional curvature or a locally product space of two spaces of constant holomorphic sectional curvature c and $-c(\geq 0)$.*

2. Proof of the main theorem

In a Kaehlerian manifold with parallel Bochner curvature tensor, we get [3]

$$\begin{aligned}(\nabla_X \bar{S})(Y,Z) &= \frac{1}{2(n+2)}\{g(X,Y)r_Z + g(X,Z)r_Y - \bar{J}(X,Y)r_{JZ} \\ &\quad -\bar{J}(X,Z)r_{JY} + 2g(Y,Z)r_X\},\end{aligned} \tag{2.1}$$

where $\bar{S}(X,Y) = g(SX,Y), \bar{J}(X,Y) = g(JX,Y)$ and $r_X = \nabla_X r$. From (2.1), we can see that

$$(n+2)(\nabla_X \bar{S})(U,U) = 2r_X g(U,U), \tag{2.2}$$

where U is a vector field given by $g(U,X) = r_X$. Then we have

Lemma 2.1. *If $\bar{S}(X,U)$ vanishes identically for any vector field X on a Kaehlerian manifold with parallel Bochner curvature tensor, then the scalar curvature is constant.*

Proof. If we differentiae $S\nabla r = 0$ and make use of (2.2), then we get the above result.

We define $S^{(2)}$, $\alpha(2)$ and $\beta(2)$ by

$$
\begin{aligned}
S^{(2)} &= SS, \\
\alpha(2) &= \text{trace } S^{(2)}, \\
\beta(2) &= g(S^{(2)}U, U).
\end{aligned} \tag{2.3}
$$

Then we can define inductively $S^{(a)}$, $\alpha(a)$ and $\beta(a)$ for any positive integer a. Obviously $\alpha(1)$ is the scalar curvature of M.

Since the Ricci operator S satisfies

$$SJ = JS, \tag{2.4}$$

we see that $S^{(a)}J = JS^{(a)}$. By use of (2.1) and (2.3), we get

$$2(n+2)(\nabla_X \bar{S})(S^{(a)}U, U) = 3\beta(a)r_X + \lambda^2 \bar{S}^{(a)}(X, U), \tag{2.5}$$

where $\bar{S}^{(a)}$ is defined by $\bar{S}^{(a)}(X, Y) = g(S^{(a)}X, Y)$ and $\lambda^2 = g(U, U)$. From the equation (2.1), we obtain

$$
\begin{aligned}
&2(n+2)(\nabla_X \bar{S})(S^{(a)}U, S^{(b)}U) \\
&= \beta(a)\bar{S}^{(b)}(X, U) + \beta(b)\bar{S}^{(a)}(X, U) + 2\beta(a+b)r_X.
\end{aligned} \tag{2.6}
$$

On the other hand, (2.1), (2.3) and (2.4) imply

$$(n+2)(\nabla_X \bar{S})(Y, S^{(a)}Y) = 2\bar{S}^{(a)}(X, U) + \alpha(a)r_X. \tag{2.7}$$

By the definitions of $\alpha(a)$ and $\bar{S}^{(a)}(X, Y)$, we can see that

$$\alpha(a+1) = \text{ trace } [S^{(a)}S]. \tag{2.8}$$

and

$$(\nabla_X \bar{S}^{(a)})(Y, SY) = a(\nabla_X \bar{S})(Y, S^{(a)}Y). \tag{2.9}$$

Therefore, by use (2.7), (2.8) and (2.9), we get

$$(n+2)(\nabla_X \alpha(a+1) = (a+1)\{2\bar{S}^{(a)}(X, U) + \alpha(a)r_X\}, \tag{2.10}$$

and that if the scalar curvature r is constant, then $\alpha(m)$ is constant for all integer $m > 1$.

Conversely, if we assume that $\alpha(m+1)$ is constant for $m > 1$, then we get

$$2S^{(m)}\nabla r + \alpha(m)\nabla r = 0 \tag{2.11}$$

and that

$$2\bar{S}^{(m)}(U,U) + \alpha(m)\lambda^2 = 0. \tag{2.12}$$

For the case where m is even, since $\bar{S}^{(m)}(U,U) \geq 0$ and $\alpha(m) \geq 0$, we see that $\alpha(m)\lambda^2 = 0$, that is, $\alpha(m) = 0$ or r is constant. Therefore, it is sufficient to show that r is constant when $\alpha(m)$ is constant for m is even. If we put $m = 2a+2$, then we obtain

$$2\bar{S}^{(2a+1)}(X,U) + \alpha(2a+2)r_X = 0 \tag{2.13}$$

by virtue of (2.10). Differentiating (2.13) covariantly and taking account of (2.13), we obtain

$$2(\nabla_X \bar{S}^{(2a+1)})(U,U) + \lambda^2 \nabla_X \alpha(2a+1) = 0,$$

which and (2.10) imply

$$2(n+2)(\nabla_X \bar{S}^{(2a+1)})(U,U) + (2a+1)\lambda^2\{2S^{(2a)}r_X + \alpha(2a)r_X = 0, \tag{2.14}$$

Taking account of the definition of $\bar{S}^{(2a+1)}(X,Y)$, we can verify that

$$\begin{aligned}(\nabla_X \bar{S}^{(2a+1)})(U,U) &= 2(\nabla_X \bar{S})(U, S^{(2a)}U) + (\nabla_X \bar{S})(S^{(a)}U, S^{(a)}U) \\ &\quad +2(\nabla_X \bar{S})\{(S^{(2a-1)}U, SU) + (S^{(2a-2)}U, S^{(2)}U) \\ &\quad + \cdots + (S^{(a+1)}U, S^{(a-1)}U)\}.\end{aligned}$$

By use of (2.5) and (2.6), it turns out to be

$$\begin{aligned}(n+2)(\nabla_X \bar{S}^{(2a+1)})(U,U) &= \lambda^2 S^{(2a)}r_X + 2(a+1)\beta(2a)r_X \\ &\quad +\beta(a)S^{(a)}r_X + \beta(1)S^{(2a-1)}r_X + \beta(2a-1)Sr_X \\ &\quad + \cdots + \beta(a-1)S^{(a+1)}r_X + \beta(a+1)S^{(a-1)}r_X,\end{aligned}$$

so (2.14) is reduced to

$$\begin{aligned}&\frac{(2a+1)}{2}\lambda^4\alpha(2a) + 4(a+1)\lambda^2\beta(2a) + \beta(a)^2 \\ &2\{\beta(1)\beta(2a-1) + \beta(2)\beta(2a-2) + \cdots + \beta(a-1)\beta(a+1)\} = 0.\end{aligned} \tag{2.15}$$

By the definition of $\beta(a)$ and simple calculations, we can see that

$$\begin{aligned}(2.16)\qquad &\lambda^2\beta(2a)+2\beta(s)\beta(2a-s)+\beta(2s)\beta(2a-2s)\\ &=\frac{1}{\lambda^2}g(T,T)+\alpha(2a-2s)g(Q,Q),\end{aligned}$$

where we have put

$$T=\lambda^2 S^{(a)}U+\beta^{(s)}S^{(a-s)}U,\quad Q=S^{(a)}U-\frac{\beta(s)}{\lambda^2}U,$$

so the right hand side of (2.16) is non-negative. By use of (2.16), the left hand side of (2.15) is divided into non-negative terms and consequently $\lambda^2=0$, that is, the scalar curvature is constant. Thus we have

Theorem 2.2. *Let M be a Kaehlerian manifold with parallel Bochner curvature tensor. Then the scalar curvature is constant if and only if $\bar{S}_{(m)}$ is constant on M for a positive integer $m(\neq 2)$.*

Moreover, Ki and Kwon [2] proved that $\alpha(2)$ is constant if and only if r is constant for the case of an indefinite manifold with Kaehler-Bochner metric. By the same method of this result, we can easily obtain

Lemma 2.3. *$\alpha(2)$ is constant if and only if r is constant on the manifold with Kaehler-Bochner metric.*

Considering Theorem 2.2 and Lemma 2.3, we get

Theorem 2.4. *The scalar curvature r is constant if and only if $\alpha(m),(m>1)$ is constant on the manifold with Kaehler-Bochner metric.*

Consequently, combining Theorem A and Theorem 2.4, we obtain the main theorem.

References

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Department of Mathematics Education, Kyungpook National University, Taegu 702-701, Korea.

Department of Mathematics, Kyung Hee University, Yongin 449-701, Korea.