Kyungpook Mathematical Journal
Volume 32, Number 2, December, 1992

WELL-CHAINED RELATOR SPACES

J. Kurdics and Árpád Száz

Introduction

In this paper, we extend four basic characterizations of well-chained uniformities of Levine [4] to those of well-chained relators.

And combining our present results with some former ones, we establish some substantial generalizations of two relevant theorems of Gaal [1, pp. 101 and 142].

The necessary prerequisites concerning relators, which are possibly unfamiliar to the reader, will be briefly laid out in the next two preparatory sections.

0. Terminology and notations

If $\mathcal{R}$ is a nonvoid family of reflexive relations R on a set X, the family $\mathcal{R}$ is called a relator on X, and the ordered pair $X(\mathcal{R}) = (X, \mathcal{R})$ is called a relator space.

If (x_α) and (y_α) are nets, A and B are sets, and x is a point in a relator space $X(\mathcal{R})$, then we write

(i) $(y_\alpha) \in Lim_{\mathcal{R}}(x_\alpha)$ $((y_\alpha) \in Adh_{\mathcal{R}}(x_\alpha))$ if $((x_\alpha, y_\alpha))$ is eventually (frequently) in each $R \in \mathcal{R}$;

(ii) $x \in \lim_{\mathcal{R}}(x_\alpha)(x \in adh_{\mathcal{R}}(x_\alpha))$ if $(x) \in Lim_{\mathcal{R}}(x_\alpha)$ $((x) \in Adh_{\mathcal{R}}(x_\alpha))$;

(iii) $B \in Cl_{\mathcal{R}}(A)$ $(B \in Int_{\mathcal{R}}(A)$ if $R(B) \cap A \neq \emptyset$ $(R(B) \subset A)$ for all (some) $R \in \mathcal{R}$;

(iv) $x \in cl_{\mathcal{R}}(A)(x \in int_{\mathcal{R}}(A))$ if $x \in Cl_{\mathcal{R}}(A)$ $(x \in Int_{\mathcal{R}}(A))$.

If $\mathcal{R}$ is a relator on X, then the relators

$$\begin{aligned} \mathcal{R}^* &= \{S \subset X^2 : \exists R \in \mathcal{R} : R \subset S\}, \\ \mathcal{R}^\sharp &= \{S \subset X^2 : \forall A \subset X : \exists R \in \mathcal{R} : R(A) \subset S(A)\}, \\ \hat{\mathcal{R}} &= \{S \subset X^2 : \forall x \in X : \exists R \in \mathcal{R} : R(x) \subset S(x)\}, \end{aligned}$$

Received November 20, 1989.

are called the uniform, proximal and topological refinements of $\mathcal{R}$, respectively.

Namely, if $\mathcal{R}$ is a relator on X, then $\mathcal{R}^*, \mathcal{R}^\sharp$ and $\hat{\mathcal{R}}$ are the largest relators on X such that $Lim_{\mathcal{R}^*} = Lim_{\mathcal{R}}(Adh_{\mathcal{R}^*} = Adh_{\mathcal{R}})$, $Cl_{\mathcal{R}^\sharp} = Cl_{\mathcal{R}}$ $(Int_{\mathcal{R}^\sharp} = Int_{\mathcal{R}})$ and $\lim_{\hat{\mathcal{R}}} = \lim_{\mathcal{R}}$ $(adh_{\hat{\mathcal{R}}} = adh_{\mathcal{R}})$ or $cl_{\hat{\mathcal{R}}} = cl_{\mathcal{R}}$ $(int_{\hat{\mathcal{R}}} = int_{\mathcal{R}})$, respectively.

Moreover, a subset A of a relator space $X(\mathcal{R})$ is called

(i) proximally closed (open) if $X \setminus A \notin Cl_{\mathcal{R}}(A)(A \in Int_{\mathcal{R}}(A))$;

(ii) topologically closed (open) if $cl_{\mathcal{R}}(A) \subset A(A \subset int_{\mathcal{R}}(A))$;

(iii) proximally (topologically) clopen if it is both proximally (topologically) closed and open.

Clearly, a proximally closed (open) set is also topologically closed (open), but the converse need not be true. Moreover, a set is proximally (topologically) closed iff its complement is proximally (topologically) open.

On the other hand, a relator $\mathcal{R}$ on X, or a relator space $X(\mathcal{R})$ is called topologically compact if for each $R \in \hat{\mathcal{R}}$ there exists a finite set $A \subset X$ such that $R(A) = X$.

Namely, a relator space $X(\mathcal{R})$ is topologically compact iff each interior cover $\mathcal{A}$ of $X(\mathcal{R})$ has a finite subcover $\mathcal{B}$, or equivalently each directed net (x_α) in $X(\mathcal{R})$ is adherent.

Finally, a relator $\mathcal{R}$ on X, or a relator space $X(\mathcal{R})$, is called

(i) uniformly directed if for each $R, S \in \mathcal{R}$ there exists a $T \in \mathcal{R}$ such that $T \subset R \cap S$;

(ii) strongly proximally directed if for any $A_i \subset X$ and $R_i \in \mathcal{R}$ with $i = 1, 2, \cdots, n$, there exists an $R \in \mathcal{R}$ such that $R(A_i) \subset R_i(A_i)$ for all $i = 1, 2, \cdots, n$.

(iii) topologically transitive if for each $x \in X$ and $R \in \mathcal{R}$ there exist $S, T \in \mathcal{R}$ such that $T(S(x)) \subset R(x)$;

(iv) proximally symmetric if for each $A \subset X$ and $R \in \mathcal{R}$ there exists an $S \in \mathcal{R}$ such that $S(A) \subset R^{-1}(A)$.

Clearly, a uniformly directed relator is also strongly proximally directed, but the converse need not be true. On the other hand, a relator $\mathcal{R}$ is proximally symmetric iff the relation $Cl_{\mathcal{R}}$ is symmetric.

1. Some basic facts on connected relators

Definition 1.1. A relator $\mathcal{R}$ on X, or a relator space $X(\mathcal{R})$ is called connected if $A^2 \cup (X \setminus A)^2 \notin \mathcal{R}$ for all proper nonvoid subset A of X.

Moreover, $\mathcal{R}$ or $X(\mathcal{R})$ is called uniformly, proximally and topologically connected if the relators $\mathcal{R}^*, \mathcal{R}^\sharp$ and $\mathcal{R}$ are connected, respectively.

The appropriateness of this definition and the validity of the next theorems have been established in [3].

Theorem 1.2. *A relator space $X(\mathcal{R})$ is proximally (topologically) connected if no proper nonvoid subset A of $X(\mathcal{R})$ is proximally (topologically) clopen.*

Theorem 1.3. *A proximally symmetric relator space $X(\mathcal{R})$ is proximally connected if no proper nonvoid subset A of $X(\mathcal{R})$ is proximally open.*

Theorem 1.4. *A proximally symmetric and topologically fine relator space $X(\mathcal{R})$ is topologically connected if and only if no proper nonvoid subset A of $X(\mathcal{R})$ is topologically open.*

Theorem 1.5. *A uniformly directed relator space $X(\mathcal{R})$ is proximally connected if and only if it is uniformly connected.*

To state a further relevant property of connected relators, we also need to the following.

Definition 1.6. A relator $\mathcal{R}$ on X is called a Lebesgue relator, and a relator space $X(\mathcal{R})$ is called a Lebesgue relator space if for each $S \in \hat{\mathcal{R}}$ there exists a function f from X into X such that $S \circ f \in \mathcal{R}$.

The appropriateness of this definition and the validity of the next theorem have been established in [8].

Theorem 1.7. *A strongly proximally directed, topologically transitive and topologically compact relator space $X(\mathcal{R})$ is a Lebesgue relator space.*

Moreover, as a particular case of a more general result, we also have

Theorem 1.8. *A Lebesgue relator space $X(\mathcal{R})$ is topologically connected if and only if it is uniformly connected.*

2. Preliminary characterizations of well-chained relators

The origin of the following definition goes back to Cantor.(See Thron[9, p.29].)

Definition 2.1. A relator $\mathcal{R}$ on X, or a relator space $X(\mathcal{R})$, will be called well-chained if for any $x, y \in X$ and $R \in \mathcal{R}$ there exists a finite family $(x_i)_{i=0}^n$ in X such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in R$ for all

$i = 0, 1, \cdots, n$.

Moreover, $\mathcal{R}$ or $X(\mathcal{R})$ will be called uniformly, proximally and topologically well-chained if the relators $\mathcal{R}^*, \mathcal{R}^\sharp$ and $\hat{\mathcal{R}}$ are well-chained, respectively.

Remark 2.2. Because of the inclusions $\mathcal{R} \subset \mathcal{R}^* \subset R^\sharp \subset \hat{\mathcal{R}}$, it is clear that 'topologically well-chained' $\Rightarrow$ 'proximally well-chained' $\Rightarrow$ 'uniformly well-chained' $\Rightarrow$ 'well-chained'.

In the sequel, we shall show that 'uniformly well-chained' and 'proximally well-chained' are actually equivalent to 'well-chained', but 'topologically well-chained' is not equivalent to 'well-chained'.

For this, we shall first extend three basic characterizations of well-chained uniformities of Levine [4] to those of well-chained relators.

Our first theorem is a straightforward extension of Levine's [4, Corollary 2.3].

Theorem 2.3. *If $X(\mathcal{R})$ is a relator space, then the following assertions are equivalent:*

(i) *$X(\mathcal{R})$ is well-chained;*

(ii) *$X^2 = \cup_{n=1}^{\infty} R^n$ for all $R \in \mathcal{R}$.*

Proof. A simple reformulation of Definition 2.1 shows that (i) holds if and only if for any $x, y \in X$ and $R \in \mathcal{R}$ there exists a positive integer n such that $(x, y) \in R^n$. And hence, the equivalence of (i) and (ii) is quite obvious.

While, our second theorem is a natural extension of Levine's [4, Theorem 2.2].

Theorem 2.4. *If $X(\mathcal{R})$ is a relator space, then the following assertions are equivalent:*

(i) *$X(\mathcal{R})$ is well-chained:*

(ii) *X^2 is the only transitive member of $\mathcal{R}^*$.*

Proof. If $S \in \mathcal{R}^*$, then there exists an $R \in \mathcal{R}$ such that $R \subset S$. Theorefore, if (i) holds and S is transitive, then by Theorem 2.3 we clearly have

$$X^2 = \cup_{n=1}^{\infty} R^n \subset \cup_{n=1}^{\infty} S^n \subset S.$$

And thus (ii) also holds.

On the other hand, if $R \in \mathcal{R}$, then it is clear that

$$S = \cup_{n=1}^{\infty} R^n$$

is a transitive relation on X such that $R \subset S$. Therefore, if (ii) holds, then we necessarily have $S = X^2$. Thus, again by Theorem 2.3, (i) also holds.

Remark 2.5. Because of the reflexivity of the elements of $\mathcal{R}$, hence we can also state that $X(\mathcal{R})$ is well-chained if and only if X^2 is the only preorder in $\mathcal{R}^*$.

3. Main characterizations of well-chained relators

Now, having Theorem 2.3 and 2.4, we can also easily prove a natural extension of Levine's [4, Corollary 2.4].

Theorem 3.1. *If $X(\mathcal{R})$ is a relator space, then the following assertions are equivalent:*

(i) *$X(\mathcal{R})$ is well-chained;*

(ii) *no proper nonvoid subset A of $X(\mathcal{R})$ is proximally open.*

Proof. If (ii) does not hold, then there exists a proper nonvoid subset A of X such that $R(A) \subset A$ for some $R \in \mathcal{R}$. Hence, it is clear that

$$(\cup_{n=1}^{\infty} R^n)(A) = \cup_{n=1}^{\infty} R^n(A) \subset A.$$

And thus, by Theorem 2.3, (i) does not also hold. Consequently, (i) implies (ii).

On the other hand, if (i) does not hold, then by Theorem 2.4, there exists a transitive relation S on X such that $R \subset S$ for some $R \in \mathcal{R}$, and $A = S(x) \neq X$ for some $x \in X$. Hence, it is clear that

$$R(A) \subset S(A) = S^2(x) = S(x) = A.$$

And thus (ii) does not also hold. Consequently, (ii) also implies (i).

Remark 3.2. Because of [6, Theorem 2.6], hence we can also state that a relator space $X(\mathcal{R})$ is well-chained if and only if no proper nonvoid subset A of $X(\mathcal{R})$ is proximally closed.

Remark 3.3. Moreover, by [6, Theorem 3.1], hence we can also state that a relator space $X(\mathcal{R})$ is well-chained if and only if for each proper nonvoid subset A of X there exists a net $((x_\alpha, y_\alpha))$ in $A \times (X \setminus A)$ such that $(x_\alpha) \in Lim_{\mathcal{R}}(y_\alpha)$. $((x_\alpha) \in Adh_{\mathcal{R}}(y_\alpha))$.

However, at the present, it is more interesting to point out that Theorem 3.1 can also be used to easily prove the next two important theorems.

Theorem 3.4. *If $X(\mathcal{R})$ is a relator space, then the following assertions are equivalent:*

(i) *$X(\mathcal{R})$ is well-chained;*

(ii) *$X(\mathcal{R})$ is uniformly well-chained;*

(iii) *$X(\mathcal{R})$ is proximally well-chained.*

Proof. By [5, Corollary 5.9], it is clear that the proximally open subsets of $X(\mathcal{R}^\sharp)$ and $X(\mathcal{R}^*)$ coincide with those of $X(\mathcal{R})$. And thus Theorem 3.1 can be applied to get the stated equivalences.

Theorem 3.5. *If $X(\mathcal{R})$ is a relator space, then the following assertions are equivalent:*

(i) *$X(\mathcal{R})$ is topologically well-chained;*

(ii) *no proper nonvoid subset A of $X(\mathcal{R})$ is topologically open.*

Proof. By [5, Theorem 6.7], it is clear that the proximally open subsets of $X(\mathcal{R})$ coincide with the topologically open subsets of $X(\mathcal{R})$. Thus, Theorem 3.1 can again be applied to get the stated equivalence.

The fact that 'topologically well-chained' is not, in general, equivalent to 'well-chained' can be at once seen from the next simple.

Example 3.6. If $X = \{1,2,3\}$ and $R_i \subset X^2$ for $i = 1,2$, such that

$$R_1(1) = \{1,2\}, \quad R_1(2) = \{2,3\}, \quad R_1(3) = \{3,1\},$$

$$R_2(1) = \{1,2\}, \quad R_2(2) = X, \qquad R_2(3) = \{3,2\},$$

then $\mathcal{R} = \{R_i\}_{i=1}^2$ is a well-chained relator on X such that $\mathcal{R}$ is not topologically well-chained.

To check this, note that $R_i^2 = X^2$ for $i = 1,2$. And moreover, if $S \subset X^2$ such that

$$S(1) = \{1,2\}, \quad S(2) = \{2,3\}, \quad S(3) = \{3,2\},$$

then $S \in \hat{\mathcal{R}}$, but $1 \notin S^n(2)$ for all positive integer n.

4. A further characterization of well-chained relators

In addition to the above theorems, using Theorem 2.4 and 3.1, we can also prove the following remarkable analogue of Levine's [4, Corollary 2.5].

Theorem 4.1. *If $X(\mathcal{R})$ is a relator space, then the following assertions are equivalent:*

(i) $X(\mathcal{R})$ *is well-chained;*

(ii) $A^2 \cup (X \setminus A) \times X \notin \mathcal{R}^*$ *for all proper nonvoid subset* A *of* X.

Proof. If A is a proper nonvoid subset of X, then it is clear that

$$S = A^2 \cup (X \setminus A) \times X$$

is a transitive relation on X such that $S = X^2$. Therefore, if (i) holds, then by Theorem 2.4, we necessarily have $S \notin \mathcal{R}^*$. And thus (ii) also holds.

On the other hand, if (i) does not hold, then by Theorem 3.1, there exists a proper nonvoid subset A of X such that $R(A) \subset A$ for some $R \in \mathcal{R}$. Hence, it is clear that

$$R \subset A^2 \cup (X \setminus A) \times X,$$

and thus (ii) does not also hold. Consequently, (ii) also implies (i).

By this theorem, it is clear that the Davis-Pervin relator [6, p.195] cannot, in general, be well-chained. More precisely, using Theorem 4.1, one can easily check the next striking

Example 4.2. If $\mathcal{A}$ is a nonvoid family of subsets of a set X and

$$\mathcal{R}_{\mathcal{A}} = \{A^2 \cup (X \setminus A) \times X : A \in \mathcal{A}\},$$

then the following assertions are equivalent:

(i) $\mathcal{R}_{\mathcal{A}}$ is well-chained;

(ii) $\mathcal{A} \subset \{\emptyset, X\}$;

(iii) $\mathcal{R}_{\mathcal{A}} = \{X^2\}$;

(iv) $\mathcal{R}_{\mathcal{A}}$ is topologically well-chained.

Remark 4.3. Using Theorem 1.3, in [3] we have proved that (i) $\mathcal{R}_{\mathcal{A}}$ is proximally connected if and only if there is no proper nonvoid subset B of X such that both B and $X \setminus B$ are in $\mathcal{A}$;

(ii) $\mathcal{R}_{\mathcal{A}}$ is topologically connected if and only if there is no proper nonvoid subset B of X such that both B and $X \setminus B$ are unions of certain members of $\mathcal{A}$.

5. Relationships with connected relators

As an immediate consequence of Theorem 1.2 and Theorems 3.1, 3.4 and 3.5, we can at once state

Theorem 5.1. *A proximally (topologically) well-chained relator space $X(\mathcal{R})$ is proximally (topologically) connected.*

Hence, by Theorem 1.8, it is clear that we also have the following useful.

Theorem 5.2. *A well-chained Lebesgue relator space $X(\mathcal{R})$ is topologically connected.*

This latter theorem, together with Theorem 1.7, at once yields a substantial extension of the 'if part' of Theorem III.3.9 of Gaal [1, p.142].

Theorem 5.3. *A strongly proximally directed, topologically transitive and topologically compact well-chained relator space $X(\mathcal{R})$ is topologically connected.*

Remark 5.4. By example 4.2 and Remark 4.3, it is clear that even a topologically connected relator space $X(\mathcal{R})$ need not be well-chained.

However, combining Theorem 1.3 with Theorems 3.1 and 3.4, we can still state an essential improvement of Theorem II.7.3 of Gaal [1, p. 101].

Theorem 5.5. *A proximally symmetric relator space $X(\mathcal{R})$ is proximally well-chained if and only if it is proximally connected.*

Hence, by Theorem 1.5 and Theorem 3.4, it is clear that we also have the following extension of Levine's [4, Corollary 2.5].

Theorem 5.6. *A uniformly directed and proximally symmetric relator space $X(\mathcal{R})$ is uniformly well-chained if and only if it is uniformly connected.*

Moreover, as an immediate consequence of Theorem 1.4 and Theorem 3.5, we can also state the following analogue of Levine's [4, Corollary 4.2].

Theorem 5.7. *A topologically fine and proximally symmetric relator space $X(\mathcal{R})$ is topologically well-chained if and only if it is topologically connected.*

Notes. Particular cases of Theorems 5.3 and 5.5 are also treated in Whyburn and Duda [10, p.37].

Moreover, a slightly incorrect particular case of Theorem 5.6 can also be found in James [2, p. 126].

References

[1] S.A. Gaal, *Point Set Topology*, Academic Press, New York, 1964.

[2] I.M. James, *Topological and Uniform Spaces*, Springer-Verlag, New York, 1987.

[3] J. Kurdics and Á. Száz, *Connected relator spaces*, preprint.

[4] N. Levine, *Well-chained uniformities*, Kyungpook Math. J. 11(1971), 143-149.

[5] Á. Száz, *Basic tools and mild continuities in relator spaces*, Acta Math. Hungar. 50(1987), 177-201.

[6] Á. Száz, *Directed, topological and transitive relators*, Publ. Math. Debrecen 35(1988), 179-196.

[7] Á. Száz, *Inverse, symmetric and neighbourhood relators*, preprint.

[8] Á. Száz, *Lebesgue relators*, preprint.

[9] W.J. Thron, *Topological Structures*, Holt, Rinehart and Winston, New York, 1966.

[10] G. Whyburn and E. Duda, *Dynamic Topology*, Springer-Verlag, New York, 1979.

Institute of Mathematics, Lajos Kossuth University, H-4010 Debrecen, Hungary.