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BITOPOLOGICAL $[a, b]$-COMPACTNESS

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1. Introduction

The theory of $[a, b]$-compactness in topological spaces dates back to the work of P. Alexandrov and P. Urysohn in the 1920's. Since then many mathematicians have contributed to the theory and to generalizations of it. Despite the large amount of work which has been done on $[a, b]$-compactness, it is still an active area today. The important concepts of compactness, countable compactness and the Lindelof property in topological spaces are all special cases of $[a, b]$-compactness and this is one reason for studying the general property. The purpose of this paper is to introduce the concept of $[a, b]$-compactness for bitopological spaces.

2. Preliminaries

Let the letters a, b, m and n denote infinite cardinal numbers with $a \leq b$ and $[a, b]$ stand for the set of all cardinal m such that $a \leq m \leq b$. Let $|E|$ denote cardinal number of a set E and m^+ denote the first cardinal strictly larger than m. A space X is called $[a, b]$-compact^r if every open cover $\mathcal{U}$ of X such that $|\mathcal{U}|$ is a regular cardinal in $[a, b]$ has a subcover $\mathcal{U}^1 \subset \mathcal{U}$ with $|\mathcal{U}^1| < |\mathcal{U}|$. This concept was introduced in [1] and the superscript 'r' is a reminder of the "restriction of regularity" in definition [5]. A space X is called $[a, b]$-compact if every open cover $\mathcal{U}$ of X with $|\mathcal{U}| \leq b$ has a subcover of cardinality strictly less than a. Further X is said to be $[a, \infty]$-compact if it is $[a, b]$-compact for all $b \geq a$. This idea was introduced in 1950 by YU. Smirnov [8]. Essentially the same property was studied independently in 1957 by I.S. Gaal [4]. The work of Gaal mentioned in this paper has been reworded to coincide with the terminology of P. Alexandrov and P. Urysohn [1].

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In section 3 the concept of $[a, b]$-compactness for bitopological spaces is introduced and its properties investigated. In section 4 we give some characterizations of such spaces.

3. Concept of $[a, b]$-compactness in bitopological spaces

Based on the three formulations of $[a, b]$-compactness for topological spaces found in [5], we define the corresponding formulations for bitopological spaces as follows:

(A) A bitopological space (X, T_1, T_2) is said to be $[a, b]$-pairwise compact if every pairwise open cover $\mathcal{U}$ of X with $|\mathcal{U}| \leq b$ has a subcover $\mathcal{V}$ such that $|\mathcal{V}| < a$.

(B) A bitopological space (X, T_1, T_2) is said to be $[a, b]$-pairwise compact^r if every open cover $\mathcal{U}$ of X whose cardinality $|\mathcal{U}| = m$ is a regular cardinal number in $[a, b]$ has a subcover of cardinality $< m$.

(C) A bitopological space (X, T_1, T_2) is said to be (m, n)-pairwise compact if from every pairwise open cover $\mathcal{U}$ of X, whose cardinality is at most n, one can select a subcover $\mathcal{V}$ of X whose cardinality is at most m.

Remarks. 1. Observe that $[\aleph_0, \infty]$-pairwise compact is the same as pairwise compact [3], $[\aleph_0, \aleph_0]$-pairwise compact is the same as pairwise countably compact [7] and $[\aleph, \infty]$-pairwise compact is the same as pairwise Lindelof [6] property.

2. Every (m, n)-pairwise compact space is $[m^+, n]$-pairwise compact.

3. It follows easily that $[a, b]$-pairwise compactness implies $[a, b]$-pairwise compactness^r. The converse is not true. For example let X be a set whose cardinality is singular, $\mathcal{D}$ be the discrete topology on X, T, any topology on X such that $T \subset \mathcal{D}$. Then $(X, \mathcal{D}, T)$ is pairwise $[a, \infty]$-compace^r but not pairwise $[a, \infty]$-compact.

4. Let R be the set of all real numbers, $\mathcal{R} = \{\emptyset, R, (a, \infty) | a \in R\}$ $\mathcal{L} = \{\emptyset, R, (-\infty, a) | a \in R\}$. Then $(R, \mathcal{R}, \mathcal{L})$ is pairwise $[\aleph_0, \infty]$-compact. However, both $(R, \mathcal{L})$ and $(R, \mathcal{R})$ are not $[\aleph_0, \infty]$-compact. This example shows that pairwise $[a, b]$-compactness of (X, T_1, T_2) need not imply $[a, b]$-compactness of (X, T_1) and (X, T_2).

4. Simple properties of $[a, b]$-pairwise compact spaces

(a) Finite union of $[a,b]$-pairwise compact subsets are $[a,b]$-pairwise compact.

(b) If Y is a T_1 or T_2 closed subset of $[a,b]$-pairwise compact space (X,T_1,T_2), then Y is an $[a,b]$-pairwise compact subspace of X.

Proof. Let $\mathcal{U}$ a pairwise open covering of Y such that $|\mathcal{U}| \leq b$. Adjoining $O_0 = X \backslash Y$ we obtain a pairwise open covering of X with cardinality at most b. By hypothesis there is a subcover say $\mathcal{V}$ of $\mathcal{U} \cup \{O_0\}$ such that $|\mathcal{V}| < a$. Discarding O_0 we obtain a subcover of $\mathcal{U}$, whose cardinality is strictly less than a.

(c) If every T_1 and T_2 open subset of (X,T_1,T_2) is $[a,b]$-pairwise compact subspace, then X is hereditarily $[a,b]$-pariwise compact.

Proof. Let Y be arbitrary subset of X and $\mathcal{U}$ be a pairwise open cover of Y such that $|\mathcal{U}| \leq b$. Then $\mathcal{U}$ is a pairwise open cover of the set $\cup_{0\in\mathcal{U}}O$. Write $\cup_{0\in\mathcal{U}}O = (\cup_{0\in\mathcal{U},0\in T_1}O) \cup (\cup_{0\in\mathcal{U},0\in T_2}O = V \cup W$ (say). If $V = \emptyset$(or $W = 0$) then $U_{0\in\mathcal{U}}O$ is T_2-open (T_1-open) and so by hypothesis there is a subfamily $\mathcal{V}$ of $\mathcal{U}$ covering $\cup_{0\in\mathcal{U}}O$ such that $|\mathcal{V}| < a$. This subfamily also covers the set Y and so Y is $[a,b]$-pairwise compact. If neither V nor W is empty, then V and W are both $[a,b]$-pairwise compact by hypothesis and so $V \cup W$ is $[a,b]$-pairwise compact and now the conclusion follows easily.

(d) Let (X,T_1,T_2) be a bitopological space and $\{Y_k\}_{k\in K}$ be a family of subsets of X. If each Y_k is $[a,b]$-pairwise compact for some $a > |K|$, then $\cup_{k\in K}Y_k$ is $[a,b]$-pairwise compact.

(e) If $X = \prod_{\alpha\in\Gamma} X_\alpha$ is $[a,b]$-pairwise compact, then every subproduct $X(K) = \prod_{k\in K} X_{\alpha_k}$ is $[a,b]$-pairwise compact. Similarly if X is hereditarily $[a,b]$-pairwise compact, then every subproduct $X(K)$ is hereditarily $[a,b]$-pairwise compact.

(f) The product of $[\aleph_0,\infty]$ -pairwise compact space X and an $[a,\infty]$-pairwise compact space Y is $[a,\infty]$-pairwise compact.

5. Characterization of $[a,b]$-pairwise compactness

The following Theorem gives some characterizations $[a,b]$-pairwise compact spaces:

Theorem. *For a bitopological space (X,T_1,T_2), the following are equivalent.*

(a) *(X,T_1,T_2) is $[a,b]$-pairwise compact.*

(b) *For each non-empty subset V in T_1, the topology $T_2(V)$ is $[a,b]$-compact and for each non-empty set V in T_2, the topology $T_1(V)$ is $[a,b]$-compact.*

(c) *Each T_1-closed proper subset of X is $[a,b]$-compact w.r.t. T_2 and each T_2-closed proper subset of X is $[a,b]$-compact w.r.t. T_1.*

Proof. (a) $\Rightarrow$ (b). Let V be any non-empty T_1 open set and let $\mathcal{U}$ be a $T_2(V)$-open cover of X such that $|\mathcal{U}| \leq b$. So $\mathcal{U} = \{V \cup U_\alpha | \alpha \in \Gamma\}$ where $U_\alpha \in T_2$ for each $\alpha \in \Gamma$. Then $\mathcal{U}^1 = \{V\} \cup \{U_\alpha | \alpha \in \Gamma\}$ is a pairwise open cover of X such that $|\mathcal{U}^1| \leq b$ and so by hypothesis it has a subcover say $\mathcal{V} = \{V\} \cup \{U_\alpha | \alpha \in \Gamma'\}$, $\Gamma' \subset \Gamma$ and $|\mathcal{V}| < a$ we add $\{\mathcal{V}\}$ to the subcover if necessary. Then $\mathcal{V}^1 = \{V \cup U_\alpha | \alpha \in \Gamma^1\}$ is the desired subcover of $\mathcal{U}$ for X, so that $T_2(V)$ is $[a,b]$-compact similarly $T_1(V)$ is $[a,b]$-compact for each non-empty set V in T_2.

(b) $\Rightarrow$ (c). Let K be any proper T_1-closed subset of X, so that $V = X - K$ is a non-empty T_1-open set. Let $\{U_\alpha | \alpha \in \Gamma\} = \mathcal{U}$ be a T_2-open cover of K such that $|\mathcal{U}| \leq b$. Then $\mathcal{U}^1 = \{V \cup U_\alpha | \alpha \in \Gamma\}$ is a $T_2(V)$-open cover of X with $|\mathcal{U}^1| \leq b$ and so we have, $X = V \cup \{\cup_{\alpha \in \Gamma'} U_\alpha\}$. $\Gamma' \subset \Gamma$ and $|\Gamma'| < a$. Then $K \subset \cup_{\alpha \in \Gamma'} U_\alpha$. So that K is T_2-$[a,b]$-compact. Similary, each T_2-closed proper subset of X is $[a,b]$-compact w.r.t. T_1.

(c) $\Rightarrow$ (a) Let $\mathcal{U}$ be a pairwise open cover of X with $|\mathcal{U}| \leq b$. Let the T_1-open sets in $\mathcal{U}$ be $\{U_\beta | \beta \in \Gamma\}$ and let the T_2 open sets in $\mathcal{U}$ be $\{V_\alpha | \alpha \in \wedge\}$. Two cases arise.

Case (i) $X = \cup_{\alpha \in \wedge} V_\alpha$. Choose a $\beta_0 \in \Gamma$ such that $U_{\beta_0} \neq \emptyset$. Then $\{V_\alpha | \alpha \in \wedge\}$ is a T_2-open cover of the T_1-closed proper subset $X - U_{\beta_0}$. So there is a subcover $\mathcal{V} = \{V_\alpha | \alpha \in \wedge' \subset \wedge\}$ such that $|\wedge'| < a$, and $X - U_{\beta_0} \subset \cup_{\alpha \in \wedge'} V_\alpha$. Then $\{U_{\beta_0}, V_\alpha\} = \mathcal{U}'$ is a subcover of $\mathcal{U}$ for X with $|\mathcal{U}'| < a$.

Case (ii) $\cup_{\alpha \in \Gamma} V_\alpha \neq X$. Then $K = X - \cup_{\alpha \in \wedge} V_\alpha$ is a proper T_1-closed subset of X and $K \subset \cup_{\beta \in \Gamma} U_\beta$. Hence there is a subcover $\{U_\beta\}_{\beta \in \Gamma' \subset \Gamma}$ such that $|\Gamma'| < a$ and $K \subset \cup_{\beta \in \Gamma'} U_\beta$. If $\cup_{\beta \in \Gamma'} U_\beta = X$, there is nothing more to prove. If $\cup_{\beta \in \Gamma'} U_\beta \neq X$, then $X - \cup_{\beta \in \Gamma'} U_\beta$ is a proper T_1-closed subset of X contained in $\cup_{\alpha \in \wedge} V_\alpha$. By hypothesis there is a subcover $\mathcal{V} = \{V_\alpha\}_{\alpha \in \wedge'}$, $|\wedge'| < a$ with $X - (\cup_{\beta \in \Gamma'} U_\beta) \subset \cup_{\alpha \in \wedge'} V_\alpha$. Then $\{U_\beta\}_{\beta \in \Gamma'} \cup \{V_\alpha\}_{\alpha \in \wedge'}$ is the required subcover of $\mathcal{U}$ (since a is infinite).

Note. The above theorem is a generalization of a result proved in [2].

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