Kyungpook Mathematical Journal
Volume 32, Number 2, December, 1992

ON A COMMUTATIVITY THEOREM OF QUADRI FOR SEMI-SIMPLE RINGS

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In this paper we prove that a semi-simple ring R in which for any x, y in R, there exist positive integers $m = m(x, y)$ and $n = n(x, y)$ such that $[x, [x^m, (xy)^n + (yx)^n]] = 0$, then R is commutative.

1. In a paper [3], M.A. Quadri and M.A. Khan proved as follows: Suppose in a semi-simple ring R, for a pair of elements x, y in R, there exist positive integers $m = m(x, y)$ and $n = n(x, y)$ such that $[x^m, (xy)^n] = [(yx)^n, x^m]$, then R is commutative. We generalize this result by taking hypothesis $[x^m, (xy)^n] - [(yx)^n, x^m] \in Z(R)$. In fact we prove the more general version of this fact.

Theorem. *Let R be a semi-simple ring, in which for any x, y in R there exist positive integers $m = m(x, y)$ and $n = n(x, y)$ such that $[x, [x^m, (xy)^n + (yx)^n]] = 0$, then R is commutative.*

In all that follows R will be an associative ring. $Z(R)$ denotes the center of R and for any pair a, b in R, $[a, b] = ab - ba$.

2. To prove the above theorem we establish the following lemmas.

Lemma 2.1. *If $[a, [a, b]] = 0$ for all $a, b \in R$ then $2[a, b]^2 = [a, [a, b^2]]$.*

Proof. It is straighforward to check.

Lemma 2.2. *Let R be a prime ring in which for any x, y in R there exist positive integers $\ell = \ell(x, y)$, $m = m(x, y)$ and $n = n(x, y)$ such that $[x^\ell, [x^m, y^n]] = 0$, then R has no non zero nilpotent element.*

Received December 15, 1990.

Proof. Let a be a non zero element of R with $a^2 = 0$. In the hypothesis, putting ax for x and xa for y and using $a^2 = 0$, we obtain

$$(ax)^{m+\ell}(xa)^n = 0 \tag{1}$$

Again putting $xa + x$ for x and using $a^2 = 0$, we get

$$(axa + ax)^{m+\ell}(xa)^n = 0.$$

Simplifying the above equation and using $a^2 = 0$, we get

$$\{(ax)^{m+\ell}a + (ax)^{m+\ell}\}(xa)^n = 0$$

Using (1), we get $(ax)^{m+\ell}a(xa)^n = 0$ which further yields

$$(ax)^{m+\ell+n+1} = 0 \text{ for all } x \in R \tag{2}$$

This implies $aR = 0$. Because if $aR \neq 0$, then the equation (2) asserts that aR is a non zero nil right ideal, satisfying the identity $(b)^{m+\ell+n+1} = 0$ for all $b \in aR$. But by Herstein's lemma [1, Lemma 1.1], R has a non zero nilpotent ideal which is a contradiction. Hence $aR = 0$ i.e. $aRa = (0)$. This forces that $a = 0$ as R is prime.

Lemma 2.3. *For elements a, b, c of a ring R.*

(1) *If $[a, b] = 0$ then $[a, [b, c]] = [b, [a, c]]$*

(2) *If $[a, b] = 0$ and $[a, [b, c]] = 0$ then $[a^m, b^n, c]] = 0$ for all positive integers m and n.*

Proof. (1) is straightforward. (2) From (1) it follows that $0 = [a^m, [b, c]] = [b, [a^m, c]]$ whence $0 = [b^n, [a^m, c]] = [a^m, [b^n, c]]$.

Lemma 2.4. *Let R be a prime ring satisfying hypothesis of lemma 2.2, then R is commutative.*

Proof. Without loss of generality, we may assume by Lemma 2.3 (2) that $\ell = m$. Then by hypothesis

$$[x^m, [x^m, y^n]] = 0. \tag{3}$$

If $Char R \neq 2$, putting y^2 for y in (3), we get

$$[x^m, [x^m, y^{2n}]] = 0. \tag{4}$$

Choose $a = x^m$, $b = y^n$ in lemma 2.1 so that hypothesis becomes $[x^m, [x^m, y^n]] = 0$. Therefore we conclude that

$$2[x^m, y^n]^2 = [x^m, [x^m, y^{2n}]] \tag{5}$$

R.H.S. of (5)= L.H.S. of (4) = 0. Hence L.H.S. of (5) = $2[x^m, y^n]^2 = 0$, since $CharR \neq 2$ we have $[x^m, y^n]^2 = 0$. This further yields by lemma 2.2, that $[x^m, y^n] = 0$. Now applying Herstein's theorem [2], we get R is commutative.

If $CharR = 2$, on simplifying (3), we obtain $[x^{2m}, y^n] = 0$. This again by Herstein's theorem [2, Theorem 1] gives that R is commutative.

Lemma 2.5. *Let R be a division ring, in which for given x, y in R, there exist positive integers $m = m(x,y)$ and $n = n(x,y)$ such that $[x,[x^m,(xy)^n + (yx)^n]] = 0$, then R is commutative.*

Proof. Let $x \neq 0$. Then by hypothesis there exist positive integers $m = m(x, x^{-1}y)$ and $n = n(x, x^{-1}y)$ such that $[x,[x^m,(x,x^{-1}y)^n + (x^{-1}yx)^n]] = 0$. This implies that $[x,[x^m, y^n + x^{-1}y^n x]] = 0$. i.e.

$$x^{m+1}y^n - xy^n x^m - x^{m-1}y^n x^2 + x^{-1}y^n x^{m+2} = 0$$

Multiplying by x on the left, we obtain

$$x^{m+2}y^n - x^2 y^n x^m - x^m y^n x^2 + y^n x^{m+2} = 0$$

Which on simplification yields

$$[x^2,[x^m,y^n]] = 0.$$

Since every division ring is prime, so by lemma 2.4 we have that R is commutative.

Proof of Theorem. First, we claim that theorem is true for primitive rings. If R is division ring (which in particular is primitive also) satisfying $[x,[x^m,(xy)^n + (yx)^n]] = 0$, then by Lemma 2.5 R is commutative. Suppose R is primitive ring which is not a division ring then it will not satisfy $[x,[x^m,(xy)^n + (yx)^n]] = 0$. For instance we note that no complete matrix ring D_r over a division ring D with $r > 1$ satisfies the identity $[x,[x^m,(xy)^n + (yx)^n]] = 0$. This can be verified by taking D_2 and elements $x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ in D_2. Hence we get a contradiction. That is theorem holds for primitive rings.

Now if R is semi simple ring then it is isomorphic to a subdirect sum of primitive rings R_α each of which is a homomorphic image of R. Note that the identity $[x,[x^m,(xy)^n + (yx)^n]] = 0$ is inherited by all subrings and all homomorphic images of R, hence R_α and subrings of R are all commutative. Therefore the semi-simple ring R is commutative.

The condition of semi simplicity can not be dropped. This can easily be seen by taking R as the ring of 3×3 strictly upper triangular matrices over an arbitrary ring. Here hypothesis $[x,[x^m,(xy)^n + (yx)^n]] = 0$ is satisfied by R yet it is not commutative.

To be precise we take $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| a,b,c,d \in Z \right\}$ which is a ring of 3×3 strictly upper triangular matrix over Z.

References

[1] I.N. Herstein, *Topics in ring theory*, University of Chicago Press, Chicago and London, 1969.

[2] ________, *A Commutativity Theorem*, J. Algebra 38(1976), 238-241.

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