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ON A CLASS OF MULTIVALENT FUNCTIONS DEFINED BY FRACTIONAL INTEGRAL

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We introduce a subclass, namely $W_{k,p}(A,B,\alpha)$ of the class $S_{k,p}$ of analytic function defined by

$$f(z) = z^p - \sum_{n=k}^{\infty} |a_{p+n}| z^{p+n} \quad (p \in N)$$

in the unit disc D. The object of the present paper is to determine sharp coefficient estimate, distortion theorems, radius of convexity and closure theorems for this class defined by fractional integral.

1. Introduction

Let $S_{k,p}$ denote the class of functions of the form

$$(1.1) \qquad f(z) = z^p - \sum_{n=k}^{\infty} |a_{p+n}| z^{p+n}, p \in N = (1,2,3,\cdots),$$

which are analytic and p-valent in the unit disc $D = \{z : |z| < 1\}$. Let $W_{k,p}(A,B)$ denote the class of those functions of $S_{k,p}$ which satisfy the condition

$$(1.2) \qquad \left|\frac{\frac{f'(z)}{z^{p-1}} - p}{B\frac{f'(z)}{z^{p-1}} - pA}\right| < 1, z \in D,$$

where $-1 \le A < B \le 1, 0 < B \le 1$.

For the subclass $W_{k,p}(A,B)$, we obtain the following result which will be used in our further study.

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Lemma 1. *A function* $f(z) = z^p - \sum_{n=k}^{\infty} |a_{p+n}| z^{p+n}$ *is in* $W_{k,p}(A, B)$ *if and only if*

$$\sum_{n=k}^{\infty} \frac{(n+p)(1+B)}{p(B-A)} |a_{p+n}| \leq 1. \tag{1.3}$$

The proof of lemma is similar to that of Vinod Kumar [3], so we omit the proof.

The function

$$f(z) = z^p - \frac{p(B-A)}{(n+p)(1+B)} z^{p+n}, n \geq K \tag{1.4}$$

is an extremal function.

Corollary 1. *If* $f \in W_{k,p}(A, B)$, *then*

$$|a_{p+n}| \leq \frac{p(B-A)}{(n+p)(1+B)},$$

equality is only for functions of the form (1.4).

There are several definitions of fractional integral. In [4], Owa defined the fractional integral as follows:

Definition. The fractional integral of order α is defined by

$$D_z^{-\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{f(\xi) d\xi}{(z-\xi)^{1-\alpha}},$$

where $0 < \alpha < 1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\xi)^{\alpha-1}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi) > 0$.

A function F belongs to the class $W_{k,p}(A, B, \alpha)$ if it satisfies

$$F(z) = \frac{\Gamma(\alpha+p+1)}{\Gamma(p+1)} z^{-\alpha} D_z^{-\alpha} f(z) \tag{1.5}$$

for some f belonging to $W_{k,p}(A, B)$. After a simple computation we have

$$F(z) = z^p - \sum_{n=k}^{\infty} \frac{\Gamma(n+p+1)\Gamma(\alpha+p+1)}{\Gamma(n+p+\alpha+1)\Gamma(p+1)} |a_{p+n}| z^{p+n}.$$

The subclasses $S_{k,p}^*(A, B, \alpha)$ and $C_{k,p}(A, B, \alpha)$ of $S_{k,p}$ obtained by replacing $f'(z)/z^{p-1}$ with $z \cdot f'(z)/f(z)$ and $[1 + z\frac{f''(z)}{f'(z)}]$ respectively in (1.2) have been studied by the authors [1].

In the present paper, our aim is to obtain a necessary and sufficient condition in terms of coefficients for a function F to be in $W_{k,p}(A,B,\alpha)$ and, consequently, we show that

$$W_{k,p}(A,B,\alpha) \subset W_{k,p}(A,B).$$

Then we extend the above inclusion relationship.

Applying the result of coefficient estimates we obtain class preserving integral operators of the form

$$G(z) = \frac{\gamma+p}{z^\gamma}\int_0^z u^{\gamma-1}F(u)du, \gamma > -p, \text{ for } W_{k,p}(A,B,\alpha).$$

Conversely when $G(z) \in W_{k,p}(A,B,\alpha)$, radius of p-valence of $F(z)$ has been determined. Further we obtain sharp results concerning distortion theorem and radius of convexity for the class $W_{k,p}(A,B,\alpha)$. It is also shown that the class $W_{k,p}(A,B,\alpha)$ is closed under "arithmetic mean" and "convex linear combinations". Also we obtain some distortion theorems for the fractional integral of the elements of $W_{k,p}(A,B)$.

Now we prove the following theorem which will be required in the next section.

Theorem 1. *A function $F(z) = z^p - \sum_{n=k}^{\infty} |b_{p+n}|z^{p+n}$ belongs to $W_{k,p}(A,B,\alpha)$ if and only if*

$$\sum_{n=k}^{\infty} |b_{p+n}|/T(n,\alpha) \leq 1. \tag{1.6}$$

where

$$T(n,\alpha) = \left[\frac{p(B-A)}{(n+p)(1+B)} \cdot \frac{\Gamma(n+p+1)\Gamma(\alpha+p+1)}{\Gamma(n+p+\alpha+1)\Gamma(p+1)}\right]$$

Proof. By definition, $F \in W_{k,p}(A,B,\alpha)$ if it satisfies the relation (1.5) for some $f \in W_{k,p}(A,B)$. Let

$$f(z) = z^p - \sum_{n=k}^{\infty} |a_{p+n}|z^{p+n}.$$

Then, from (1.5) we obtain

$$F(z) = z^p - \sum_{n=k}^{\infty} |b_{p+n}|z^{p+n}.$$

where

$$|b_{p+n}| = \frac{\Gamma(n+p+1)\Gamma(\alpha+p+1)}{\Gamma(n+p+\alpha+1)\Gamma(p+1)}|a_{p+n}|$$

or

$$|a_{p+n}| = \frac{\Gamma(n+p+\alpha+1)\Gamma(p+1)}{\Gamma(n+p+1)\Gamma(\alpha+p+1)}|b_{p+n}|, n \geq k.$$

With the help of (1.3), the required result follows.

The function

$$F(z) = z^p - T(n,\alpha)z^{p+n} \tag{1.7}$$

is an extremal function

Corollary 2. *If $F \in W_{k,p}(A,B,\alpha)$, then*

$$|b_{p+n}| \leq T(n,\alpha),$$

with equality only for the functions of the form (1.7).

2. Main results

Let $F \in W_{k,p}(A,B,\alpha)$. Then

$$F(z) = z^p - \sum_{n=k}^{\infty} |b_{p+n}|z^{p+n},$$

where

$$|b_{p+n}| = \frac{\Gamma(n+p+1)\Gamma(\alpha+p+1)}{\Gamma(n+p+\alpha+1)\Gamma(p+1)}|a_{p+n}|.$$

Clearly

$$\frac{\Gamma(n+p+1)\Gamma(\alpha+p+1)}{\Gamma(n+p+\alpha+1)\Gamma(p+1)} = \prod_{j=1}^{n} \frac{p+j}{p+\alpha+j} < 1, \text{ for all } \alpha > 0.$$

Thus $|b_{p+n}| < |a_{p+n}|$, for all $n \geq k$, and therefore

$$\begin{aligned}\sum_{n=k}^{\infty} \frac{(n+p)(1+B)}{p(B-A)}|b_{p+n}| &< \sum_{n=k}^{\infty} \frac{(n+p)(1+B)}{p(B-A)}|a_{p+n}| \\ &\leq 1, \text{ since } f \in W_{K,p}(A,B).\end{aligned}$$

Hence $F \in W_{k,p}(A,B)$, and thus we get the inclusion relation

$$W_{k,p}(A,B,\alpha) \subset W_{k,p}(A,B). \tag{2.1}$$

Since $\lim_{\beta\to 0} W_{k,p}(A,B,\beta) = W_{k,p}(A,B)$. The relation (2.1) is equivalent to

$$W_{k,p}(A,B,\alpha) \subset \lim_{\beta\to 0} W_{k,p}(A,B,\beta).$$

The following theorem is an extension of the above relation.

Theorem 2. *If $0 < \beta \le \alpha$, then*

$$W_{k,p}(A,B,\alpha) \subset W_{k,p}(A,B,\beta).$$

Proof. Let $F(z) = z^p - \sum_{n=k}^{\infty} |b_{p+n}| z^{p+n}$ belongs to $W_{k,p}(A,B,\alpha)$. Then from (1.6), we have

$$\sum_{n=k}^{\infty} |b_{p+n}|/T(n,\alpha) \le 1. \tag{2.2}$$

Since $\beta < \alpha$, we have

$$\begin{aligned}\frac{\Gamma(n+p+\beta+1)\Gamma(p+1)}{\Gamma(n+p+1)\Gamma(\beta+p+1)} &= \prod_{j=1}^{n} \frac{p+\beta+j}{p+j} \\ &< \prod_{j=1}^{n} \frac{p+\alpha+j}{p+j} \\ &= \frac{\Gamma(n+p+\alpha+1)\Gamma(p+1)}{\Gamma(n+p+1)\Gamma(\alpha+p+1)}.\end{aligned}$$

Therefore

$$\sum_{n=k}^{\infty} |b_{p+n}|/T(n,\beta) \le \sum_{n=k}^{\infty} |b_{p+n}|/T(n,\alpha). \tag{2.3}$$

Using (2.2) in (2.3) we obtain

$$\sum_{n=k}^{\infty} |b_{p+n}|/T(n,\beta) \le 1.$$

Hence $F \in W_{K,p}(A,B,\beta)$.

Now we study class preserving integral operator for $W_{k,p}(A,B,\alpha)$.

Theorem 3. *Let γ be a real number such that $\gamma > -p$. If $F \in W_{k,p}(A,B,\alpha)$, then the function G defined by*

$$G(z) = \frac{\gamma+p}{z^\gamma} \int_0^z u^{\gamma-1} F(u)\,du \tag{2.4}$$

is also an element of $W_{k,p}(A,B,\alpha)$.

Proof. Let $F(z)=z^p-\sum_{n=k}^{\infty}|b_{p+n}|z^{p+n}$, then $G(z)=z^p-\sum_{n=k}^{\infty}|C_{p+n}|z^{p+n}$, where

$$|C_{p+n}|=\frac{\gamma+p}{\gamma+p+n}|b_{p+n}|<|b_{p+n}|.$$

Therefore

$$\sum_{n=k}^{\infty}\frac{|C_{p+n}|}{T(n,\alpha)}<\sum_{n=k}^{\infty}\frac{|b_{p+n}|}{T(n,\alpha)}\leq 1.$$

Hence $G\in W_{k,p}(A,B,\alpha)$.

Following theorem is the converse problem of the above theorem.

Theorem 4. *Let* γ *be a real number such that* $\gamma>-p$. *If* $G(z)\in W_{k,p}(A,B,\alpha)$, *then the function* F *defined in (2.4) is p-valent in* $|z|<R^*$, *where*

$$R^*=Inf_{n\geq k}[(\frac{\gamma+p}{\gamma+p+n})(\frac{p}{p+n})\frac{1}{T(n,\alpha)}]^{1/n}.$$

The result is sharp.

Proof. Let $G(z)=z^p-\sum_{n=k}^{\infty}|b_{p+n}|z^{p+n}$. It follows from (2.4) that

$$F(z)=z^p-\sum_{n=k}^{\infty}(\frac{\gamma+p+n}{\gamma+p})|b_{p+n}|z^{p+n}.$$

In order to establish the required result it sufficies to prove that

$$|\frac{F'(z)}{z^{p-1}}-p|<p \text{ for } |z|<R^*.$$

Now

$$\begin{aligned}|\frac{F'(z)}{z^{p-1}}-p| &= |-\sum_{n=k}^{\infty}(n+p)(\frac{\gamma+p+n}{\gamma+p})|b_{p+n}|z^n|\\ &\leq \sum_{n=k}^{\infty}(n+p)(\frac{\gamma+p+n}{\gamma+p})|b_{p+n}||z|^n.\end{aligned}$$

Thus $|\frac{F'(z)}{z^{p-1}}-p|<p$ if

$$\sum_{n=k}^{\infty}(\frac{n+p}{p})(\frac{\gamma+p+n}{\gamma+p})|b_{p+n}||z|^n<1. \tag{2.5}$$

Since $G \in W_{K,p}(A, B, \alpha)$, then

$$\sum_{n=k}^{\infty} |b_{p+n}|/T(n,\alpha) \le 1.$$

Therefore (2.5) will be satisfied if

$$(\frac{n+p}{p})(\frac{\gamma+p+n}{\gamma+p})|b_{p+n}||z|^n \le |b_{p+n}|/T(n,\alpha), \text{ for each } n \ge k,$$

or if

$$|z| \le [(\frac{p}{n+p})(\frac{\gamma+p}{\gamma+p+n})\frac{1}{T(n,\alpha)}]^{1/n}, \text{ for each } n \ge K.$$

Hence F is p-valent in $|z| < R^*$.

The result is sharp with extremal function

$$G(z) = z^p - T(n,\alpha)z^{p+n}, n \ge k.$$

Theorem 5. *If $F \in W_{k,p}(A, B, \alpha)$, then F is p-valently convex in the disc $|z| < R^{**}$, where*

$$R^{**} = Inf_{n \ge k}[(\frac{1}{n+p})^2 \frac{1}{T(n,\alpha)}]^{1/n}.$$

The result is sharp.

Proof. It sufficies to prove that

$$|[1 + zF''(z)/F'(z)] - p| < p \text{ for } |z| < R^{**}.$$

we have

$$\begin{aligned} |[1 + zF''(z)/F'(z)] - p| &= \left|\frac{-\sum_{n=k}^{\infty} n(n+p)|b_{p+n}|z^n}{p - \sum_{n=k}^{\infty}(n+p)|b_{p+n}|z^n}\right| \\ &\le \frac{\sum_{n=k}^{\infty} n(n+p)|b_{p+n}||z|^n}{p - \sum_{n=k}^{\infty}(n+p)|b_{p+n}||z|^n} \end{aligned}$$

Thus $|[1 + z\frac{F''(z)}{F'(z)}] - p| < p$ if

$$\sum_{n=k}^{\infty}(\frac{n+p}{p})^2|b_{p+n}||z|^n < 1. \tag{2.6}$$

Since $F \in W_{k,p}(A, B, \alpha)$, then

$$\sum_{n=k}^{\infty} |b_{p+n}|/T(n,\alpha) \leq 1.$$

Therefore (2.6) will be satisfied if

$$(\frac{n+p}{p})^2 |b_{p+n}||z|^n \leq |b_{p+n}|/T(n,\alpha); \text{ for each } n \geq k.$$

or if

$$|z| \leq [(\frac{p}{p+n})^2 \frac{1}{T(n,\alpha)}]^{1/n}, \text{ for each } n \geq k.$$

Hence F is p-Valently convex in $|z| < R^{**}$.

The result is sharp with extremal function

$$F(z) = z^p - T(n,\alpha) z^{p+n}, n \geq k.$$

Theorem 6. *Let a function $f(z) = z^p - \sum_{n=k}^{\infty} |a_{p+n}| z^{p+n}$ be in the class $W_{k,p}(A, B, \alpha)$, then we have*

$$(2.7)\quad \frac{\Gamma(p+1)}{\Gamma(p+\alpha+1)} r^{p+\alpha}[1 - T(n,\alpha) r^k] \leq |D_z^{-\alpha} f(z)| \leq \frac{\Gamma(p+1)}{\Gamma(p+\alpha+1)} r^{p+\alpha}[1 + T(n,\alpha) r^k],$$

the bounds are sharp.

Proof. Let $f(z) = z^p - \sum_{n=k}^{\infty} |a_{p+n}| z^{p+n}$. Then it follows from lemma 1,

$$\frac{(k+p)(1+B)}{p(B-A)} \sum_{n=k}^{\infty} |a_{p+n}| \leq \sum_{n=k}^{\infty} \frac{(n+p)(1+B)}{p(B-A)} |a_{p+n}| \leq 1.$$

Therefore

$$(2.8)\quad \sum_{n=k}^{\infty} |a_{p+n}| \leq \frac{p(B-A)}{(k+p)(1+B)}.$$

Let us consider the function

$$F(z) = \frac{\Gamma(p+\alpha+1)}{\Gamma(p+1)} z^{-\alpha} D_z^{-\alpha} f(z) = z^p - \sum_{n=k}^{\infty} \frac{\Gamma(p+n+1)\Gamma(\alpha+p+1)}{\Gamma(p+n+\alpha+1)\Gamma(p+1)} |a_{p+n}| z^{p+n}.$$

Then, by using (2.8), we get

$$\begin{aligned} |F(z)| &\leq r^p + \sum_{n=k}^{\infty} \frac{\Gamma(p+n+1)\Gamma(\alpha+p+1)}{\Gamma(p+n+\alpha+1)\Gamma(p+1)} |a_{p+n}| r^{p+n} \\ &\leq r^p + T(k,\alpha) r^{p+k} \end{aligned}$$

and

$$\begin{aligned} |F(z)| &\geq r^p - \sum_{n=k}^{\infty} \frac{\Gamma(p+n+1)\Gamma(\alpha+p+1)}{\Gamma(p+n+\alpha+1)\Gamma(p+1)} |a_{p+n}| r^{p+n} \\ &\geq r^p - T(k,\alpha) r^{p+k}. \end{aligned}$$

The required inequalities follows at once.

To establish the sharpness of the bounds in (2.7), we take

$$f(z) = z^p - \frac{p(B-A)}{(k+p)(1+B)} z^{p+k}.$$

In (2.7), the left hand side equality is obtained at $z = r$, whereas, the right side equality is attained at $z = re^{i\pi/k}$. Hence the bounds are sharp. This completes the proof of the theorem.

Theorem 7. *If $F \in W_{k,p}(A, B, \alpha)$ and $|z| = r$, then*

$$r^p[1 - T(k,\alpha)r^k] \leq |F(z)| \leq r^p[1 + T(k,\alpha)r^k] \tag{2.9}$$

and

$$\begin{aligned} r^{p-1}[p - (p+k)\cdot T(k,\alpha)r^k] &\leq |F'(z)| \\ &\leq r^{p-1}[p - (p+k)\cdot T(k,\alpha)r^k]. \end{aligned} \tag{2.10}$$

All these inequalities are sharp.

Proof. Let $F(z) = z^p - \sum_{n=k}^{\infty} |b_{p+n}| z^{p+n}$. Then, in the view of theorem 1.

$$\frac{1}{T(k,\alpha)} \sum_{n=k}^{\infty} |b_{p+n}| \leq \sum_{n=k}^{\infty} |b_{p+n}|/T(n,\alpha) \leq 1,$$

we have

$$\sum_{n=k}^{\infty} |b_{p+n}| \leq T(k,\alpha).$$

Now

$$\begin{aligned} |F(z)| &\leq r^p + \sum_{n=k}^{\infty} |b_{p+n}| r^{p+n} \\ &\leq r^p + T(k,\alpha) r^{p+k} \end{aligned}$$

and

$$\begin{aligned} |F(z)| &\geq r^p - \sum_{n=k}^{\infty} |b_{p+n}| r^{p+n} \\ &\geq r^p - T(k,\alpha) r^{p+k}. \end{aligned}$$

Hence (2.9) follows. Further

$$\begin{aligned} |F'(z)| &\leq pr^{p-1} + \sum_{n=k}^{\infty} |b_{p+n}|(p+n) r^{p+n-1} \\ &\leq pr^{p-1} + r^{p+k-1} \sum_{n=k}^{\infty} (p+n)|b_{p+n}| \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} |F'(z)| &\geq pr^{p-1} - \sum_{n=k}^{\infty} |b_{p+n}|(p+n) r^{p+n-1} \\ &\geq pr^{p-1} - r^{p+k-1} \sum_{n=k}^{\infty} (p+n)|b_{p+n}|. \end{aligned} \tag{2.12}$$

Since

$$\frac{1}{(p+k)T(k,\alpha)} \sum_{n=k}^{\infty} (p+n)|b_{p+n}| \leq \sum_{n=k}^{\infty} |b_{p+n}|/T(n,\alpha) \leq 1.$$

We have

$$\sum_{n=k}^{\infty} (p+n)|b_{p+n}| \leq (p+k)T(k,\alpha). \tag{2.13}$$

The inequalities in (2.10) follows by using (2.13) in (2.11) and (2.12).

Equalities are obtained in (2.9) and (2.10) by taking

$$F(z) = z^p - T(k,\alpha) z^{p+k}.$$

We note that for the above defined function F, equalities on the left hand side of (2.9) and (2.10) are obtained at $z = r$, whereas, the equality on the right hand side is attained at $z = re^{i\pi/k}$.

Lastly we show that the class $W_{k,p}(A,B,\alpha)$ is closed under "arithmetic mean" and "convex linear combination".

Theorem 8. *Let* $F_j(z) = z^p - \sum_{n=k}^{\infty} |a_{p+n}^j| z^{p+n}$, $j = 1,2,\cdots,m$. *If* $F_j \in W_{k,p}(A,B,\alpha)$ *for each* $j = 1,2,\cdots,m$, *then the function* $H(z) = z^p - \sum_{n=k}^{\infty} |c_{p+n}| z^{p+n}$, *where* $|c_{p+n}| = \frac{1}{m}\sum_{j=1}^{m} |a_{p+n}^j|$, *also belongs to* $W_{k,p}(A,B,\alpha)$.

Proof. Since $F_j \in W_{k,p}(A,B,\alpha)$, then

$$\sum_{n=k}^{\infty} |a_{p+n}^j|/T(n,\alpha) \le 1 \text{ for each } j = 1,2,\cdots,m.$$

Therefore

$$\begin{aligned}\sum_{n=k}^{\infty} |c_{p+n}|/T(n,\alpha) &= \sum_{n=k}^{\infty}[\frac{1}{m}\sum_{j=1}^{m} |a_{p+n}^j|/T(n,\alpha)] \\ &= \frac{1}{m}\sum_{j=1}^{m}[\sum_{n=k}^{\infty} |a_{p+n}^j|/T(n,\alpha)] \\ &\le 1.\end{aligned}$$

Which implies that $H(z) \in W_{k,p}(A,B,\alpha)$.

Theorem 9. *Let* $F_p(z) = z^p$ *and* $F_{p+n}(z) = z^p - T(n,\alpha)z^{p+n}$, $(n = k, k+1, \cdots)$. *Then* $F \in W_{k,p}(A,B,\alpha)$ *if and only if it can be expressed in the form*

$$F(z) = \mu_p F_p(z) + \sum_{n=k}^{\infty} \mu_{p+n} F_{p+n}(z), \tag{2.14}$$

where $\mu_{p+n} \ge 0$ *and* $\mu_p + \sum_{n=k}^{\infty} \mu_{p+n} = 1$.

Proof. Suppose that $F(z)$ can be expressed as in (2.14). Then

$$\begin{aligned}F(z) &= \mu_p F_p(z) + \sum_{n=k}^{\infty} \mu_{p+n} F_{p+n}(z) \\ &= z^p - \sum_{n=k}^{\infty} \mu_{p+n} \cdot T(n,\alpha) z^{p+n}.\end{aligned}$$

Now

$$\sum_{n=k}^{\infty} \frac{1}{T(n,\alpha)} \cdot T(n,\alpha) \cdot \mu_{p+n} = \sum_{n=k}^{\infty} \mu_{p+n} \le 1.$$

Hence by Theorem 1, $F \in W_{k,p}(A,B,\alpha)$.

Conversely, suppose that $F \in W_{k,p}(A, B, \alpha)$ and

$$F(z) = z^p - \sum_{n=k}^{\infty} |b_{p+n}| z^{p+n} \tag{2.15}$$

Setting

$$\mu_{p+n} = |b_{p+n}|/T(n, \alpha), (n = k, k+1, \cdots),$$

and

$$\mu_p = 1 - \sum_{n=k}^{\infty} \mu_{p+n},$$

from (2.15) we have

$$F(z) = \mu_p F_p(z) + \sum_{n=k}^{\infty} \mu_{p+n} F_{p+n}(z).$$

This completes the proof of theorem.

Remarks. It is worth mentioning here that

1. If we put $\alpha = 0$ and $p = 1$ in above theorems we obtain results of Vinod Kumar [3].
2. If $\alpha = 0$ and $k = 1$ in above theorems we obtain the results of S.L. Shukla and Dashrath [5].
3. If $\alpha = 0$, $p = 1$, $k = 1$, $B = \beta$ and $A = (2\alpha - 1)\beta$ where $0 < \beta < 1$, $0 \le \alpha < 1$, in above theorems we get the the results of Gupta and Jain [2].

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