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A MIXED PROBLEM FOR A SECOND ORDER STURM-LIOUVILLE EQUATION

El-Sayed A. M. A.

The Sturm-Liouville equation $(a(t)u'(t))' + b(t)u(t) = f(t)$ has been considered in several works with different kinds of the coefficients $a(t), b(t)$ and f. For example see [3] and [4]. In [4] Ralph, Harris and Kwong studied the weighted means and oscillation conditions for this equation with $a(t)$ and $b(t)$ are $n \times n$ real symmetric matrix where $a(t)$ is positive definite with $a^{-1}(t)$ is defined and $f = 0$. In [3] the asymptotic behaviour of the solution of this equation is studied where $a(t)$ is a positive and continuously differentiable function and $b(t)$ is a continuous one. Here we are concerned with a mixed problem of the Sturm-Liouville equation with operator coefficients. The existence and uniqueness of the solution are proved and some properties of the solution are investigated.

1. Introduction

Let D be a bounded region in R^n with boundary ∂D and $I = [0, T]$. Consider now the mixed problem

$$\frac{\partial}{\partial t}(A(t)\frac{\partial u(x,t)}{\partial t} + B(t)u(x,t) = 0, x \in D, 0 < t \leq T \tag{1.1}$$

$$u(x,0) = u_0(x), \quad x \in D \tag{1.2}$$

$$\frac{\partial u(x,0)}{\partial t} = 0, \quad x \in D \tag{1.3}$$

$$\sum_{|\alpha|\leq j} b_{j\alpha}(x,t)D^\alpha u(x,t) = 0, x \in \partial D, t \in I, \tag{1.4}$$

$$j = 1, 2, \cdots, m$$

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with the following assumptions

(1) $(B(t), t \in I)$ is a family of bounded linear operators defined on $L_p(D)$, strongly continuous in $t \in I$ and satisfies for $f \in L_p(D)$

$$\|B(t)f\| \leq b\|f\|, \tag{1.5}$$

where b is a positive constant.

(2) The operator $A(t)$ is strongly elliptic (uniformly in t)

$$A(t) = A(x,t,d) = \sum_{|\alpha|\leq 2m} a_\alpha(x,t)D^\alpha \tag{1.6}$$

where for each $t \in I$, the coefficients of the highest order derivatives are continuous in D, and the other coefficients are bounded and measurable in D. Also every coefficients is assumed to satisfy Holder's condition.

(3) $b_{j\alpha}(x,t) \in C^{2m-j}(\partial D)$.

Our purpose here is to obtain results concerning exsitence, uniqueness and other properties of the solution of the mixed problem (1.1), (1.2), (1.3) and (1.4) under the above conditions.

2. Existence and uniqueness

Now from the properties of the operators $A(t)$ and $B(t)$, and the results of [5] (Chap 5) we can prove the following theorem.

Theorem 2.1. *If $u_0 \in W_p^{2m}(D)$, then there exists one and only one solution*

$$u(x,t) \in W_p^{2m}(D), \ \textit{and} \ \frac{\partial u(x,t)}{\partial t} \in W_p^{2m}(D)$$

of the mixed problem (1.1), (1.2), (1.3) and (1.4).

Proof. Firstly consider the mixed problem for the equation.

$$A(t)\frac{\partial u(x,t)}{\partial t} = V(x,t), x \in D, 0 < t \leq T \tag{2.1}$$

with the initial and boundary data (1.2) and (1.4), where $V(x,t)$ is continuous in $t \in I$ with values in the Banach space $L_p(D)$, then from the properties of $A(t)$ and the results of [5], (chap 5), the solution of the mixed problem (2.1), (1.2) and (1.4) satisfies

$$\frac{\partial u(x,t)}{\partial t} = A^{-1}(t)V(x,t) \in W_p^{2m}(D) \tag{2.2}$$

and is given by

$$u(x,t) = u_0 + \int_0^t A^{-1}(s)V(x,s)ds \in W_p^{2m}(D) \tag{2.3}$$

which can be written as

$$u(x,t) = u_0 + R(t)V(x,t) \tag{2.4}$$

where $A^{-1}(t)$ is strongly continuously differentiable operator

$$A^{-1}(t) : L_p(D) \to W_p^{2m}(D)$$

and

$$\|A^{-1}(t)f\|_W \leq a\|f\|, f \in L_p(D) \tag{2.5}$$

where $\|u(x,t)\|_W = \sum_{|\alpha|\leq 2m} \|D^\alpha u(x,t)\|$ is the norm of the Banach space $W_p^{2m}(D)$, and a is a positive constant.

Now for the existence of $V(x,t)$, substitute from (2.1) into (1.1) and (1.3) to get the initial value problem

$$\frac{\partial V(x,t)}{\partial t} = -B(t)u_0 - B(t)R(t)V(x,t) \tag{2.6}$$

$$V(x,0) = 0. \tag{2.7}$$

Since

$$\begin{aligned} \|-B(t)R(t)V(x,t)\| &\leq b\int_0^t \|A^{-1}(s)V(x,s)\|ds \\ &\leq abt \max_{t\in I} \|V(x,t)\| \end{aligned} \tag{2.8}$$

i.e.

$$|||-B(t)R(t)V(x,t)||| \leq abT|||V(x,t)|||$$

where $|||V(x,t)||| = \max_{t\in I} \|V(x,t)\|$, it follows that, [2], $-B(t)R(t)$ is bounded in the Banach space $C(L_p(D), I)$ of continuous functions in $t \in I$, with values in $L_p(D)$, and hence the solution of the initial value problem (2.5) and (2.6) is given by

$$V(x,t) = -\int_0^t B(s)u_0 ds - \int_0^t B(s)R(s)V(x,s)ds \in L_p(D) \tag{2.9}$$

and continuous in $t \in I$, which completes the prove.

Corollary 2.1. *If $f(x,t) \in C(L_p(D), I)$ and $u_0 \in W_p^{2m}(D)$, then there exists one and only one solution of the mixed problem of the equation.*

$$\frac{\partial}{\partial t}(A(t)\frac{\partial u(x,t)}{\partial t}) + B(t)u(x,t) = f(x,t),$$

with the mixed data (1.2), (1.3) and (1.4).

3. Properties of the solution

Here we will prove some energy inequalities for the solution of the mixed problem (1.1), (1.2), (1.3) and (1.4).

Theorem 3.1. *Let $T^2 < \frac{2}{ab}$. If the solution of the mixed problem (1.1), (1.2), (1.3) and (1.4) exists, then it satisfies*

$$\|u(x,t)\|_w \leq \frac{(2+abT^2)}{(2-abT^2)}\|u_0\|_W \tag{3.1}$$

$$\|\frac{\partial u(x,t)}{\partial t}\|_w \leq \frac{(2abT)}{(2-abT^2)}\|u_0\|_W \tag{3.2}$$

Proof. From (2.8) and (2.10), we have

$$\|V(x,t)\| \leq bt\|u_0\| + \frac{abt^2}{2}|||V(x,t)|||$$

hence

$$|||V(x,t)||| \leq \frac{2bT}{(2-abT^2)}\|u_0\| \tag{3.3}$$

provided that

$$T^2 < \frac{2}{ab}.$$

Now from (2.3) and (3.3), we have

$$\begin{aligned}\|u(x,t)\|_W &\leq \|u_0\|_W + aT\frac{2bT}{(2-abT^2)}\|u_0\| \\ &\leq \|u_0\|_W + \frac{2abT^2}{(2-abT^2)}\|u_0\|_W\end{aligned}$$

i.e.

$$\|u(x,t)\|_W \leq \frac{(2+abT^2)}{(2-abT^2)}\|u_0\|_W.$$

Also from (2.2) and (2.3) we get

$$\begin{aligned}\|\frac{\partial u(x,t)}{\partial t}\|_W &\le \|A^{-1}(t)V(x,t)\|_W \le a\|V(x,t)\| \\ &\le a|||V(x,t)||| \le \frac{2abT}{(2-abT^2)}\|u_0\| \\ &\le \frac{2abT}{(2-abT^2)}\|u_0\|_W\end{aligned}$$

which completes the prove.

Equation (3.1) proves that the solution of the mixed problem is continuously depends on the initial data.

Now from the Sobolev's Embedding theorem [1] and our results here, we have

Corollary 3.1. *If* $2mp > n, T^2 < \frac{2}{ab}$, *and the conditions of theorems (2.1) and (3.1) are satisfied, then the solution of the mixed problem (1.1), (1.2), (1.3) and (1.4) is equivalent to a function of* $C(D)$ *and the derivative* $\frac{\partial u(x,t)}{\partial t}$ *exists in the usual sense.*

Example. The previous results can be applied to the mixed problem of the integro-differential equation

$$A(t)\frac{\partial u(x,t)}{\partial t} + \int_0^t B(s)u(x,s)ds = g(x,t)$$
$$x \in D, 0 < t \le T$$

with the mixed data (1.2) and (1.4), where $g(x,t) \in L_p(D)$ and continuous in $t \in I$.

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MATHEMATICS DEPARTMENT, FACULTY OF SCIENCE, ALEXANDRIA UNIVERSITY, ALEXANDRIA, EGYPT.