Kyungpook Mathematical Journal
Volume 32, Number 1, June, 1992

ON $*$-PRIMES AND $*$-VALUATIONS

Ismail M. Idris

We study the relations between $*$-primes and $*$-valuations to deduce a necessary and sufficient condition for extending $*$-valuations.

1. Introduction

Let $(D,*)$ be a $*$-field; that is, a skew field with an involution $*$ (an anti-automorphism of order 2). For general valuation theory on skew fields one can refer to [7]. For $*$-fields we need our valuations to also be compatible with the involution $*$. Following Holland [4], we define a $*$-valuation on a $*$-field $(D,*)$ to be a valuation w onto an additively written ordered group with the additional property that $w(x^*) = w(x)$ for all non-zero $x \in D$. One of the properties any reasonable generalization of the concept of a valuation should have is that valuations allow extensions to larger fields. In [3] and [8] a necessary and sufficient condition is given for extending an abelian valuation from a division ring D to the over division ring E. We will solve here a $*$-version of this problem were valuations are replaced with $*$-valuations. This is done by using a characterization of those $*$-primes giving rise to $*$-valuations, together with an extension theorem for $*$-primes. Basic properties of $*$-primes in a $*$-ring R are given in section (2).

2. $*$-primes

Throughout this section R will be an arbitrary $*$-ring with unit. A couple (P,R') is said to be a $*$-prime in the $*$-ring R if the following conditions are satisfied :

1) R' is a $*$-closed subring of R.

2) P is a $*$-closed prime ideal in R'.

Received February 5, 1990.

3) if $xR'y \subset P$ with $x, y \in R$ then $x \in P$ or $y \in P$.
If P is a $*$-closed prime ideal of R, then (P, R) is a $*$-prime in R.

Let T be a subset of the $*$-ring R. A subset S of R is an m-system for T iff $0 \notin S$ and for any $s_1, s_2 \in S$ there is an $x \in T$ such that $s_1 x s_2 \in S$.

Lemma 1. *If (P, R') is a $*$-prime in R then $R - P$ is an m-system for R'. Conversely, if P is a $*$-closed additive subgroup of R' which is multiplicatively closed and such that $R - P$ is an m-system for $R^p = \{r \in R | rP \subset P$ and $Pr \subset P\}$ then (P, R') is a $*$-prime in R.*

Proof. This is immediate.

Consider $S = \{(P, R') | R'$ a $*$-closed subring of R, P a $*$-closed prime ideal of $R'\}$. Let (P_1, R_1) and (P_2, R_2) be elements of S. Say that (P_1, R_1) dominates (P_2, R_2) (notation : $(P_2, R_2) < (P_1, R_1)$) iff $R_1 \supset R_2$ and $P_1 \cap R_2 = P_2$. If (P, R') is maximal in S with respect to $<$ then call it a dominating pair in R.

Lemma 2. *Let (P, R^p) be a dominating pair in R. If R' is a $*$-closed subring of R, I a $*$-closed ideal in R' such that $R^p \subset R'$ and $I \cap R^p = P$ then $I = P$ and $R' = R^p$.*

Proof. Let $T = \{(I, R') | I$ a $*$-closed ideal in R^p, $R^p \subset R'$ and $I \cap R' = P\}$. Since T is not empty it contains (by Zorn's lemma) a maximal element, say (Q, B). One can prove that Q is a $*$-closed prime ideal in B. This yields that $(Q, B) > (P, R^p)$. But (P, R^p) is a domainating pair, hence $P = Q$, $B = R^p$ follows, i.e. $T = \{(P, R^p)\}$ which proves the lemma.

By a $*$-R-ring A we mean a $*$-ring A where R is assumed to be a subring of A.

Lemma 3. *Let $\pi = (P, R')$ be an arbitrary $*$-prime in R then (K, A^k) is a $*$-prime in A which restricts to π (i.e. $K \cap R = P$ and $A^k \cap R = R'$) if and only if*

1) *K is a $*$-closed left and right R'-module.*
2) *$K \cap R = P$.*
3) *$A - K$ is an m-system for A^k.*

Proof. If (K, A^k) is a $*$-prime in A which restricts to π, then (1) and (2) are evident, and (3) holds by using Lemma (1). The converse is also true by using Lemma (1).

If S, T are subsets of A then $S < T >$ stands for $\{x \in A | x = \sum s_i t_i$,

$s_i \in S, t_i \in \overline{T}\}$ where $\overline{T}$ is the multiplicative closed set generated by T.

Theorem 4. *Let A a $*$-R-algebra, $\pi = (P, R^p)$ a fixed dominating $*$-prime in R. Let B be a $*$-closed subset of A and M a $*$-closed subset of B satisfy the following properties:*

(i) *$BP \subset P < B >$ and $BR^p \subset R^p < B >$.*

(ii) *$P < B > \cap R = P$.*

(iii) *M is an m-system for B.*

(iv) *$R^p - P \subset M$.*

(v) *$M \cap P < B > = \phi$.*

Then there is a $*$-prime (K, A') in A which restricts to π, such that $BK \subset K$ and $KB \subset K$.

One can adapt the proof of Theorem (2.3) in [8] to prove Theorem (4).

3. $*$-valuations in $*$-fields

Let $(D, *)$ be a $*$-field, and let D^x be the multiplicative group of non-zero elements of D. Following Holland [4], a function w from D^x onto an additively written ordered group Γ is called a *$*$-valuation* of D if

(i) $w(xy) = w(x) + w(y)$, for every $x, y \in D^x$.

(ii) $w(x + y) \geq \min(w(x), w(y)), x + y \neq 0$.

(iii) $w(x^*) = w(x)$.

It then follows that Γ is abelian since $w(x) + w(y) = w(xy) = w(y^* x^*) = w(y) + w(x)$.

First, we recall some basic facts about $*$-valuations in $*$-fields.

Definition. Let R be a subring of D.

(1) R is called *total* if for every $x \in D^x$, x or $x^{-1} \in R$.

(2) R is called *symmetric* if it contains $x^* x^{-1}$ for every $x \in D^x$.

(3) R is called *$*$-valuation ring* if it is total and symmetric.

Remark 5 [4]. If R is a symmetric subring of D then R is $*$-closed and preserved under conjugation.

Remark 6. If R is a $*$-closed total subring which is preserved under conjugation then R is symmetric and so it is a $*$-valuation subring.

Lemma 7[4]. *Let w be a $*$-valuation of a $*$-field D, then*

(1) *$V = \{x \in D | w(x) \geq 0\}$ is $*$-closed subring of D and $P = \{x \in D | w(x) > 0\}$ is a $*$-closed maximal ideal of V.*

(2) *V is total.*

(3) *Every ideal in V is two-sided.*

(4) *The ideal P is the unique maximal ideal of V, formed by the non-units in V, and V/P is a $*$-skew-field.*

(5) *V is symmetric (therefore preserved under conjugation).*

Proposition 8 [4]. *Given a $*$-valuation subring V of the $*$-field D, then there exist an ordered abelian group Γ and a $*$-valuation $w : D^x \to \Gamma$ such that V concides with the $*$-valuation ring of W.*

In a $*$-field which is finite-dimensional over its centre, $*$-valuation rings may be defined without demanding symmetry.

Theorem 9. *Let D be a $*$-field finite dimensional over its centre. Then any $*$-closed total subring V of D is a $*$-valuation subring.*

One can adopt the proof of Theorem (3) in [2] to prove theorem (9).

The following proposition characterises those $*$-primes in $*$-fields which yield $*$-valuation rings.

Proposition 10. *A $*$-prime (P, D^p) such that D^p is preserved under conjugation, yields a $*$-valuation of D with $*$-valuation ring D^p and maximal ideal P. Conversely, if V is a $*$-valuation ring, P its maximal ideal, then (P, V) is a $*$-prime of D.*

Proof. We first claim that D^p is total. Suppose $x^{-1} \notin D^p$ then, since $xD^px^{-1} \subset D^p$, $PxD^px^{-1} \subset P$ follows, but this yields $Px \subset P$ (using the defining property of $*$-primes). On the other hand also $xP \subset P$. Hence $x \in D^p$ and D^p is total. Clearly D^p is $*$-closed. Then, by Remark (6), D^p is a $*$-valuation ring. Now, if $x \notin P$ then as before one can show that $x^{-1} \in D^p$, a contradiction, so $x \in P$. This proves that P is maximal ideal in D^p.

For the converse, it is enough to check property (3) in the definition of $*$-primes: take $x, y \in D$ such that $xy \in P$, if $x \notin P$ then $x^{-1} \in V$ and so $x^{-1}xy = y \in P$.

Theorem 11. *Let D, E be $*$-fields, $D \subset E$, a $*$-valuation w on D extends to a $*$-valuation of E if and only if PE^c is a proper ideal in VE^c, where V is the $*$-valuation subring of w, P its maximal ideal, and E^c is the commutator subgroup of E.*

Proof. We first note that $PE^c = E^cP$ and $VE^c = E^cV$ (for, $r(xyx^{-1}y^{-1}) = ((rxy)r(rxy)^{-1}r^{-1})r$). Let $B = VE^c$ and $M = V - P$. Clearly $M \subset B \subset$

E. Since both V and E^c are ∗-closed, it follows that $(VE^c)^* = (E^c)^* V^* = E^c V = VE^c$, that is B is ∗-closed. Also, from the fact that $w(x^*) = w(x)$, it follows that M is ∗-closed. By Proposition (10), (P, V) is a ∗-prime in D. Applying Theorem (4), with $B = VE^c$ and $M = V - P$ which satisfy properties (i)-(v), yields a ∗-prime (P', V') in E such that $VE^c \subset E^{p'}$ and $P' \cap D = P$. Clearly $(P', E^{p'})$ is a ∗-prime in E.

To show that $E^{p'}$ is a ∗-valuation ring in E with maximal ideal P', one must show that $E^{p'}$ is preserved under conjugation (Proposition (10)). Let $x \in E$, and $e \in E^{p'}$, then $xex^{-1}e^{-1} \in E^c \subset E^{p'}$, so that $xex^{-1} \in E^{p'}$. Now, by Proposition (8), $E^{p'}$ defines the desired extension.

To prove the converse, assume that w_1 is an extension of w and PE^c is not a proper ideal in VE^c. Then an equation of the form

$$\sum_i a_i c_i = 1, \quad a_i \in P, \quad c_i \in E^c$$

holds. Thus $0 = w_1(1) \geq \min_i\{w(a_i) + w_1(c_i)\}$. Since $a_i \in P, w(a_i) > 0$ follows. Also $w_1(c_i) = 0$ (for, c_i is a product of commutators), so $w(a_i) + w_1(c_i) > 0$ a contradiction. Therefore PE^c is a proper ideal in VE^c.

For examples of a ∗-field extension where the condition in Theorem (11) does not hold, see [8].

There is a remarkable relation between ∗-valuations and the notion of ordering of a ∗-field D. Beginning with a definition of Baer, at least four different notions of orderings have been proposed for D[1,4, 5,6]. The connection between orderings and ∗-valuation is provided by the fact that, to any ordering $\geq$ on D, we can associate a ∗-valuation ring V (the order subring) consists of elements of D which are bounded by some rational numbers with respect to $\geq$. This has been done for the c-ording [1], the strong ordering [5], and the Jorden ordering [6]. For the notion of Baer ordering [4], it is shown that V is a total subring, but whether or not V is a ∗-valuation ring is still on open question. The following corollary of Theorem (9) is a partial answer to that question.

Corollary. *If D is a Baer ordered ∗-field which is finite dimensional over its centre, then the order subring V associated with the ordering is a ∗-valuation ring.*

References

[1] M. Chacron, *c-orderable division rings with involution*, J. of Algebra, Vol.75, No.2 (1982), pp.495-521.

[2] P.M. Cohn, *On extending valuations in division algebras*, Studia Scientiarum Mathematicarum Hungarica 16(1981), 65-70.

[3] P.M. Cohn and M. Mahdavi-Hezavehi, *Extension of valuations on skew fields*, Lecture notes in Mathematics, 825, Springer-Verlag, Berlin, 1980, 28-41.

[4] S.S. Holland (Jr.), **-valuations and ordered *-fields*, Trans. Amer. Math. Soc. 262, no.1, 1980, p.219-243.

[5] S.S. Holland (Jr.), *Strong ordering of *-field*, J. of Algebra 101(1986), p.16.

[6] I.M. Idris, *Jordan ordering of a division ring with involution*, Arbian Journal for Science and Engineering, Volume 14, Number 4 (1989), p.527-535.

[7] O.G.S. Shilling, *The theory of valuations*, Mathematical Surveys No. 4(1950).

[8] J. Van Geel, *Places and Valuations in non-commutative ring theory*, Dekker Lecture Notes Vol.71, 1981.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, AIN-SHAMS UNIVERSITY, CAIRO, EGYPT.