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SOME COMMUTATIVITY THEOREMS OF NON-ASSOCIATIVE RINGS

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Firstly, we have generalized a result of Ashraf and Quadri of associative ring for non-associative ring. Secondly, a result of Foster that a Boolean-like ring is commutative, is generalized for non-associative ring.

1. Introduction

Asharf and Quadri [1], in their paper proved that an assocative ring R with unity satisfying $[xy - x^m y^n, x] = 0$. For all $x, y \in R$ fixed integers $m > 1, n > 1$, is commutative. We generalize this result by choosing $m = n = 2$ and R a non-associative ring with unity satisfying the condition $xy - x^2y^2 \in Z(R)$ and show that such a ring turns commutative. By a non-associative ring, we mean a ring $(R, +, \circ)$ which need not satisfy the associative property with repect to $\circ$, i.e. $(R, \circ)$ is closed but need not be associative.

Next in [3], Yaqub proved that an associative Boolean-like ring is commutative. By Boolean-like ring we mean a ring satisfying the condition $(xy)^2 - x^2y - xy^2 + xy = 0$ and $2x = 0$ for all $x,\ y \in R$. We generalize Foster's result for non-associative ring by assuming the condition that $(xy)^2 - x^2y - xy^2 + xy \in Z(R)$, where $Z(R)$ denotes the centre of R.

2. We prove following theorems

Theorem 2.1. *The semisimple non-associative ring with unity 1 satisfying the condition $xy - x^2y^2 \in Z(R)$, is commutative.*

Proof. By hypothesis, we have

$$xy - x^2y^2 \in Z(R) \tag{1}$$

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Replace x by $x+1$, to obtain

$$(y - y^2 - 2xy^2) \in Z(R) \tag{2}$$

Case I : If char $R = 2$ then (2) gives,

$$y - y^2 \in Z(R)$$

Replacing y by $x+y$, in the above equation, we obtain

$$xy + yx \in Z(R).$$

Since char $R = 2$, therefore $xy - yx \in Z(R)$. Now if $Z(R) = 0$ or $Z(R) = R$ then R is essentially a commutative ring. Therefore suppose $0 \neq Z(R) \neq R$. In first instance we take R to be a simple ring. Consider the principal ideal $(xy - yx)R$, since $Z(R) \neq R$ and R is simple, so $(xy - yx)R = 0$. Now if R is division ring then $xy - yx = 0$ i.e. R is commutative. If R is simple ring which is not a division ring then R is homomorphic to D_2, the complete matrix ring of 2×2 matrices over a division ring D which must satisfy the condition $xy - x^2y^2 \in Z(R)$. In fact, if we choose $x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ then this condition fails. Hence R is commutative, if we choose R to be simple. Now semisimple ring is subdirect sum of simple rings, each of which is shown to be commutative. Hence semisimple ring R is also commutative under the given hypothesis.
Case II. Suppose char $R \neq 2$ then from (2),

$$(y - y^2 - 2xy^2)z = z(y - y^2 - 2xy^2) \quad \forall z \in R. \tag{3}$$

Replace y by $y+1$, to obtain

$$[(y+1) - (y+1)^2 - 2x(y+1)^2]z = z[(y+1) - (y+1)^2 - 2x(y+1)^2] \tag{4}$$

Simplifying (4) and using (3), we obtain

$$2yz + 4(xy)z + 2xz = 2zy + 4z(xy) + 2zx \tag{5}$$

As char $R \neq 2$ and R is semisimple,

$$yz + 2(xy)z + xz = zy + 2z(xy) + zx \tag{6}$$

Replacing x by $x+1$ in (6) and using (6), we obtain

$$\begin{aligned} 2yz &= 2zy \\ 2(yz-zy) &= 0 \end{aligned}$$

As char $R \neq 2$, we get

$$yz = zy.$$

Thus R is commutative.

Theorem 2.2. *If R is non-associative semi-simple ring with unity 1 satisfying the condition $(xy)^2 - x^2y - xy^2 + xy \in Z(R)$, then R is commutative.*

Proof. By hypothesis

$$(xy)^2 - x^2y - xy^2 + xy \in Z(R) \tag{7}$$

replace x by $x+1$ in (7), to obtain

$$[(xy)y + y(xy) - 2xy] \in Z(R) \tag{8}$$

Suppose char $R = 2$ then (8) gives

$$(xy)y + y(xy) \in Z(R)$$

since char $R = 2$, we have

$$(xy)y - y(xy) \in Z(R) \tag{9}$$

Replace y by $y+1$ in (9) and using (9) to get

$$(xy - yx) \in Z(R).$$

Applying same arguments as in the theorem 2.1, we find that R is commutative. If $char R \neq 2$, then replacing x by $x+1$ in (8) and using (8), we obtain

$$2(y^2 - y) \in Z(R)$$

As char $R \neq 2$, we get

$$y^2 - y \in Z(R)$$

$$(y^2 - y)z = z(y^2 - y) \quad \forall z \in R \tag{10}$$

Now replace y by $y+1$ in (10), to obtain

$$2yz = 2zy$$

i.e. $2(yz - zy) = 0$
As char $R \neq 2$, we get

$$yz = zy.$$

Thus R is commutative.

References

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