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ON PRODUCTS OF CONJUGATE EP_r MATRICES

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In this paper we answer the question of when product of conjugate EP_r (con-EP_r) matrices is con-EP_r.

1. Introduction

Throughout this paper we deal with complex square matrices. Any matrix A is said to be con-EP if $R(A) = R(A^T)$ or equivalently $N(A) = N(A^T)$ or equivalently $AA^+ = \overline{A^+A}$ and is said to be con-EP_r if A is con-EP and $rk(A) = r$, where $R(A), N(A)$, $\bar{A}$, A^T and $rk(A)$ denote the range space, null space, conjugate, transpose and rank of A respectively [3]. A^+ denotes the Moore-Penrose inverse of A satisfying the following four equations:

$$(1)\ AXA = A,\ (2)\ XAX = X,\ (3)\ (AX)^* = AX,\ (4)\ (XA)^* = XA\ [2].$$

A^* is the conjugate transpose of A. In general product of two con-EP_r matrices need not be con-EP_r. For instance, $\begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix}$ are con-EP_1 matrices, but the product is not con-EP_1 matrix.

The purpose of this paper is to answer the question of when the product of con-EP_r matrices is con-EP_r, analogous to that of EP_r matrices studied by Baskett and Katz [1]. We shall make use of the following results on range space, rank and generalized inverse of a matrix.

$$(1)\ R(A) \ = \ R(B) \Leftrightarrow AA^+ = BB^+$$

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$$\begin{aligned}(2)\ R(A^+) &= R(A^*)\\ (3)\ rk(A) &= rk(A^+) = rk(A^T) = rk(\bar{A})\\ (4)\ (A^+)^+ &= A.\end{aligned}$$

Results :

Theorem 1. *Let A_1 and $A_n (n > 1)$ be con-EP_r matrices and let $A = A_1 A_2 \cdots A_n$. Then the following statements are equivalent.*

(i) *A is con-EP_r.*

(ii) *$R(A_1) = R(A_n)$ and $rk(A) = r$*

(iii) *$R(A_1^*) = R(A_n^*)$ and $rk(A) = r$*

(iv) *A^+ is con-EP_r.*

Proof. (i) $\Leftrightarrow$ (ii) : Since $R(A) \subseteq R(A_1)$ and $rk(A) = rk(A_1)$. We get $R(A) = R(A_1)$. Similarly, $R(A^T) = R(A_n^T)$. Now,

$$\begin{aligned} A \text{ is con-}EP_r &\iff R(A) = R(A^T) \text{ and } rk(A) = r\\ &\qquad \text{(by definition of con-}EP_r)\\ &\iff R(A_1) = R(A_n^T) \quad \& \quad rk(A) = r\\ &\iff R(A_1) = R(A_n) \quad \& \quad rk(A) = r\\ &\qquad \text{(since } A_n \text{ is con-}EP_r) \end{aligned}$$

(ii) $\iff$ (iii) :

$$\begin{aligned} R(A_1) = R(A_n) &\iff A_1 A_1^+ = A_n A_n^+ \text{ (by result (1))}\\ &\iff \overline{A_1 A_1^+} = \overline{A_n A_n^+}\\ &\iff A_1^+ A_1 = A_n^+ A_n \text{(since } A_1, A_n \text{ are con-}EP_r)\\ &\iff R(A_1^+) = R(A_n^+) \text{ (by results (1) \& (4))}\\ &\iff R(A_1^*) = R(A_n^*) \text{ (by results (2)).} \end{aligned}$$

Therefore,

$$R(A_1) = R(A_n) \text{ and } rk(A) = r \Leftrightarrow R(A_1^*) = R(A_n^*) \text{ and } rk(A) = r.$$

(iv) $\iff$ (i) :

$$A^+ \text{ is con-}EP_r \iff R(A^+) = R(A^+)^T \text{ and } rk(A^+) = r$$

$$\begin{aligned} & \text{(by definition of con} - EP_r\text{)} \\ \Longleftrightarrow\ & R(A^+) = R(\bar{A}) \text{ and } rk(A^+) = r \\ \Longleftrightarrow\ & R(A^T) = R(A) \text{ and } rk(A) = r \\ & \text{(by results (2) and (3))} \\ \Longleftrightarrow\ & A \text{ is con-}EP_r. \end{aligned}$$

Hence the Theorem.

Corollary 1. *Let A and B be con-EP_r matrices. Then AB is a con-EP_r matrix $\Leftrightarrow rk(AB) = r$ and $R(A) = R(B)$.*

Proof. Proof follows from Theorem 1 for the product of two matrices A, B.

Remark 1. In the above corollary both the conditions that $rk(AB) = r$ and $R(A) = R(B)$ are essential for a product of two con-EP_r matrices to be con-EP_r. This can be seen in the following:

Example 1. Let $A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$, $B = \begin{bmatrix} -i & 1 \\ 1 & i \end{bmatrix}$ be con-EP_1 matrices. Here $R(A) = R(B)$, $rk(AB) \neq 1$ and AB is not con-EP_1.

Example 2. Let $A = \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} i & i \\ i & i \end{bmatrix}$ be con-EP_1 matrices. Here $R(A) \neq R(B)$, $rk(AB) = 1$ and AB is not con-EP_1.

Remark 2. In particular for $A = B$, Corollary 1 reduces to the following.

Corollary 2. *Let A be con-EP_r. Then A^k is con-$EP_r \Leftrightarrow rk(A^k) = r$.*

Theorem 2. *Let $rk(AB) = rk(B) = r_1$ and $rk(BA) = rk(A) = r_2$. If AB, B are con-EP_{r_1} and A is con-EP_{r_2}, then BA is con-EP_{r_2}.*

Proof. Since $rk(BA) = rk(A) = r_2$, it is enough to show that $N(BA) = N(BA)^T$. $N(A) \subseteq N(BA)$ and $rk(BA) = rk(A)$ implies $N(BA) = N(A)$. Similarly, $N(AB) = N(B)$. Now,

$$\begin{aligned} N(BA) &= N(A) \\ &= N(A^T) \qquad \text{(Since } A \text{ is con-}EP_{r_2}\text{)} \\ &\subseteq N(B^TA^T) \\ &= N((AB)^T) \\ &= N(AB) \qquad \text{(Since } AB \text{ is con-}EP_{r_1}\text{)} \\ &= N(B) \qquad \text{(Since } N(AB) = N(B)\text{)} \end{aligned}$$

$$= N(B^T) \quad \text{(Since } B \text{ is con-}EP_{r_1}\text{)}$$
$$\subseteq N(A^T B^T) = N((BA)^T).$$

Further, $rk(BA) = rk(BA)^T$ implies $N(BA) = N(BA)^T$. Hence the Theorem.

Lemma 1. *If A, B are con-EP_r matrices and AB has rank r, then BA has rank r.*

Proof. $rk(AB) = rk(B) - \dim\,(N(A) \cap N(B^*)^\perp)$. Since $rk(AB) = rk(B) = r$, $N(A) \cap N(B^*)^\perp = 0$.

$$\begin{aligned} N(A) \cap N(B^*)^\perp = 0 \;&\Rightarrow\; N(A) \cap N(\bar{B})^\perp = 0 \\ &\qquad \text{(Since } B \text{ is con-}EP_r\text{)} \\ &\Rightarrow\; N(\bar{A})^\perp \cap N(B) = 0 \\ &\Rightarrow\; N(A^*)^\perp \cap N(B) = 0 \\ &\qquad \text{(Since } A \text{ is con-}EP_r\text{)} \end{aligned}$$

Now,

$$rk(BA) = rk(A) - \dim(N(B) \cap N(A^*)^\perp) = r - 0 = r.$$

Hence the Lemma.

Theorem 3. *If A, B and AB are con-EP_r matrices, then BA is con-EP_r.*

Proof. Since A, B are con-EP_r matrices and $rk(AB) = r$, by Lemma 1, $rk(BA) = r$. Now the result follows from Theorem 2, for $r_1 = r_2 = r$.

Remark 3. For any two con-EP_r matrices A and B, since AB, $\overline{AB}$, $\overline{A^+}B$, $A\overline{B^+}$, A^+B^+, B^+A^+ all have the same rank, the property of a matrix being con-EP_r is preserved for its conjugate and Moore-Penrose inverse, by applying Corollary 1 for a pair of con-EP_r matrices among A, B, A^+, B^+, $\bar{A}, \bar{B}, \overline{A^+}, \overline{B^+}$ and using the result 2, we can deduce the following.

Corollary 3. *Let A, B be con-EP_r matrices. Then the following statements are equivalent.*

(i) *AB is con-EP_r.*
(ii) *$\overline{AB}$ is con-EP_r.*
(iii) *$\overline{A^+}B$ is con-EP_r.*
(iv) *$A\overline{B^+}$ is con-EP_r.*
(v) *A^+B^+ is con-EP_r.*

(vi) B^+A^+ *is con-*EP_r.

Theorem 4. *If* A, B *are con-*EP_r *matrices,* $R(\bar{A}) = R(B)$ *then* $(AB)^+ = B^+A^+$.

Proof. Since A is con-EP_r and $R(\bar{A}) = R(B)$, we have $R(A^+) = R(B)$.

That is, given $x \in C_n$ (the set of all $n \times 1$ complex matrices) there exists a $y \in C_n$ such that $Bx = A^+y$. Now,

$$Bx = A^+y \Rightarrow B^+A^+ABx = B^+A^+AA^+y = B^+A^+y = B^+Bx.$$

Since B^+B is hermitian, it follows that B^+A^+AB is hermitian. Similarly, $R(A^+) = R(B)$ implies ABB^+A^+ is hermitian.

Further by result (1), $A^+A = BB^+$. Hence,

$$\begin{aligned} AB(B^+A^+)AB &= ABB^+(BB^+)B \\ &= AB \end{aligned}$$

$$\begin{aligned} (B^+A^+)AB(B^+A^+) &= B^+(BB^+)BB^+A^+ \\ &= B^+A^+. \end{aligned}$$

Thus B^+A^+ satisfies the defining equations of the Moore-Penrose inverse, that is, $(AB)^+ = B^+A^+$. Hence the Theorem.

Remark 4. In the above Theorem, the condition that $R(\bar{A}) = R(B)$ is essential.

Example 3. Let $A = \begin{bmatrix} i & i \\ i & i \end{bmatrix}$ and $B = \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}$. Here A and B are con-EP_1 matrices, $rk(AB) = 1$, $R(\bar{A}) \neq R(B)$ and $(AB)^+ \neq B^+A^+$.

Remark 5. The converse of Theorem 4, need not be true in general. For,

Let $A = \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix}$. A and B are con-EP_1 matrices, such that $(AB)^+ = B^+A^+$, but $R(\bar{A}) \neq R(B)$.

Next to establish the validity of the converse of the Theorem 4, under certain condition, first let us prove a Lemma.

Lemma 2. *Let* $A = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$ *be an* $n \times n$ *con-*EP_r *matrix where* E *is an* $r \times r$ *matrix and if* $[EF]$ *has rank* r, *then* E *is nonsingular. Moreover there is an* $(n-r) \times r$ *matrix* K *such that* $A = \begin{bmatrix} E & EK^T \\ KE & KEK^T \end{bmatrix}$.

Proof. Since A is con-EP_r, $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is con-EP_r and $[E\ F]$ has rank r, the product $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} E & F \\ 0 & 0 \end{bmatrix}$ is a product of con-EP_r matrices which has rank r. Therefore by Lemma 1 the product $\begin{bmatrix} E & F \\ G & H \end{bmatrix}$ $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} E & 0 \\ G & 0 \end{bmatrix}$ has rank r. Hence there is an $(n-r) \times r$ matrix K and an $r \times (n-r)$ matrix L such that $G = KE$, $F = EL$, and E is nonsingular.

Therefore,

$$A = \begin{bmatrix} E & EL \\ KE & KEL \end{bmatrix}.$$

Now, set $C = \begin{bmatrix} I_r & 0 \\ -K & I_{n-r} \end{bmatrix}$ and consider

$$\begin{aligned} CAC^T &= \begin{bmatrix} I_r & 0 \\ -K & I_{n-r} \end{bmatrix} \begin{bmatrix} E & EL \\ KE & KEL \end{bmatrix} \begin{bmatrix} I_r & -K^T \\ 0 & I_{n-r} \end{bmatrix} \\ &= \begin{bmatrix} E & EL \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_r & -K^T \\ 0 & I_{n-r} \end{bmatrix} = \begin{bmatrix} E & -EK^T + EL \\ 0 & 0 \end{bmatrix} \end{aligned}$$

CAC^T is con-EP_r. From $N(A) = N(CAC^T)$ it follows that $EL - EK^T = 0$, and so $L = K^T$, completing the proof.

Theorem 5. *If A, B are con-EP_r matrices, $rk(AB) = r$ and $(AB)^+ = B^+A^+$, then $R(\bar{A}) = R(B)$.*

Proof. Since A is con-EP_r, by Theorem 3 in [3], there is a unitary matrix U such that, $U^T AU = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$, where D is $r \times r$ nonsingular matrix.

Set $U^* B\bar{U} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$.

$$\begin{aligned} U^T AB\bar{U} = U^T AUU^* B\bar{U} &= \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \begin{bmatrix} DB_1 & DB_2 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} D & 0 \\ 0 & I_{n-r} \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} \text{ has rank } r \text{ and thus,} \end{aligned}$$

$$U^* BAU = U^* B\bar{U}U^T AU = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} B_1 D & 0 \\ B_3 D & 0 \end{bmatrix}$$

$$= \begin{bmatrix} B_1 & 0 \\ B_3 & 0 \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & I_{n-r} \end{bmatrix}$$

has rank r. It follows that $\begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} B_1 & 0 \\ B_3 & 0 \end{bmatrix}$ have rank r, so that B_1 is nonsingular.

By Lemma 2, $U^*B\bar{U} = \begin{bmatrix} B_1 & B_1K^T \\ KB_1 & KB_1K^T \end{bmatrix}$, with $rk(U^*B\bar{U}) = rk(B_1) = r$. By using Penrose representation for the generalized inverse [4], we get

$$(U^*B\bar{U})^+ = \begin{bmatrix} B_1^*PB_1^* & B_1^*PB_1^*K^* \\ \bar{K}B_1^*PB_1^* & \bar{K}B_1^*PB_1^*K^* \end{bmatrix}$$

where $P = (B_1B_1^* + B_1K^T\bar{K}B_1^*)^{-1}B_1(B_1^*B_1 + B_1^*K^*KB_1)^{-1}$

$$U^TB^+U = (U^*B\bar{U})^+ = \begin{bmatrix} Q & QK^* \\ \bar{K}Q & \bar{K}QK^* \end{bmatrix} \text{ where}$$

$$Q = (I + K^T\bar{K})^{-1}B_1^{-1}(I + K^*K)^{-1}$$

$$U^*A^+\bar{U} = (U^TAU)^+ = \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\begin{aligned} U^TAB\bar{U} &= U^TAB\bar{U}(U^TAB\bar{U})^+U^TAB\bar{U} \\ &= U^TAB\bar{U}(U^T(AB)^+\bar{U})U^TAB\bar{U}(\text{ since } U \text{ is unitary}) \\ &= U^TAB\bar{U}(U^TB^+A^+\bar{U})U^TAB\bar{U} \text{ (by hypothesis)} \\ &= U^TAB\bar{U}(U^TB^+U)(U^*A^+\bar{U})U^TAB\bar{U} \text{ (since } U \text{ is unitary)} . \end{aligned}$$

On simplification, we get,

$$\begin{aligned} DB_1QB_1 + DB_2\bar{K}QB_1 &= DB_1 \\ \Rightarrow DB_1(I + B_1^{-1}B_2\bar{K})QB_1 &= DB_1. \end{aligned}$$

Since $B_2 = B_1K^T, QB_1 = (I + K^T\bar{K})^{-1}$. Hence $(I + K^T\bar{K}) = (QB_1)^{-1} = I$. Thus $K^T\bar{K} = 0$ which implies $K^*K = 0$ so that $K = 0$.

$$U^*B\bar{U} = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$U^TAU = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow U^*\bar{A}\bar{U} = \begin{bmatrix} \bar{D} & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $\bar{D}$ and B_1 are $r \times r$ nonsingular matrices we have

$$\begin{aligned} R(\bar{D}) = R(B_1) &\Rightarrow R\left(\begin{bmatrix} \bar{D} & 0 \\ 0 & 0 \end{bmatrix}\right) = R\left(\begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}\right) \\ &\Rightarrow R(U^*\bar{A}\bar{U}) = R(U^*B\bar{U}) \\ &\Rightarrow R(\bar{A}) = R(B). \end{aligned}$$

Hence the Theorem.

Theorem 6. *Let A, B are con-EP_r matrices, $rk(AB) = r$ and $(AB)^+ = A^+B^+$, then AB is con-EP_r.*

Proof.

$$\begin{aligned} R(B) &= R(B^T) \quad (\text{since } B \text{ is con-}EP_r) \\ \Rightarrow R(\bar{B}) &= R(B^*) \\ &= R(B^*A^*) \quad (\text{since } R(B^*A^*) \subseteq R(B^*) \\ &\qquad \text{and } rk(AB)^* = rk(AB) = r = rk(B^*)) \\ &= R(AB)^* = R(AB)^+ \text{ (by result (2))} \\ &= R(A^+B^+) \text{ (by hypothesis)} \\ &\subseteq R(A^+) = R(A^*) = R(\bar{A}) \\ &\qquad \text{(by result (2) \&} A \text{ is con-}EP_r). \\ \Rightarrow R(\bar{B}) &= R(\bar{A}) \Rightarrow R(B) = R(A). \end{aligned}$$

Since $rk(AB) = r$ and $R(B) = R(A)$, by Corollary 1, AB is con-EP_r. Hence the Theorem.

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