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ON A CLASS OF MULTIVALENT SPIRAL-LIKE FUNCTIONS

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Let $S_p^\lambda(A,B,q)$ denote the class of functions $f(z)=z^p+\sum_{n=1}^{\infty}a_{n+p}z^{n+p}$ which are regular in the unit disc $U=\{z:|z|<1\}$ and satisfy the condition

$$e^{i\lambda}\frac{zf'(z)}{f(z)}<\frac{p+\{pB+(A-B)(p-q)\}z}{1+Bz}\cos\lambda+ip\sin\lambda, z\in U,$$

and the class $T_p^\lambda(A,B,q)$ denote the class of functions $g(z)=\frac{1}{z^p}+\sum_{n=0}^{\infty}a_nz^{n-p+1}$ regular in the punctured disc $U'=\{z:0<|z|<1\}$ and satisfy the condition

$$-e^{i\lambda}\frac{zg'(z)}{g(z)}<\frac{p+\{pB+(A-B)(p-q)\}z}{1+Bz}\cos\lambda+ip\sin\lambda, z\in U',$$

where A,B are arbitrary fixed numbers $-1\le B<A\le 1$, $\lambda\in(-\pi/2,\pi/2)$ and $0\le q<p$.

In this paper we obtain sharp coefficient estimates for the class $S_p^\lambda(A,B,q)$ and $T_p^\lambda(A,B,q)$ and maximization of $|a_{p+2}-\mu a_{p+1}^2|$ over the class $S_p^\lambda(A,B,q)$ for real and complex values of μ.

1. Introduction

Let $S_p^\lambda(p\ge 1)$ denote the class of the form

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{n+p}z^{n+p}$$

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which are regular and p-valent in the unit disc $U = \{z : |z| < 1\}$. For A, B fixed, $-1 \leq B < A \leq 1$, $\lambda \in (-\pi/2, \pi/2)$ and $0 \leq q < p$, we say that $f \in S_p^\lambda(A, B, q)$ if

$$e^{i\lambda}\frac{zf'(z)}{f(z)} < \frac{p + \{pB + (A-B)(p-q)\}z}{1 + Bz}\cos\lambda + ip\sin\lambda, z \in U.$$

It follows from the definition of subordination that

$$e^{i\lambda}\frac{zf'(z)}{f(z)} = \frac{p + \{pB + (A-B)(p-q)\}w(z)}{1 + Bw(z)}\cos\lambda + ip\sin\lambda, z \in U, \quad (1.1)$$

where $w(z)$ is regular in U and satisfying the conditions $w(0) = 0$, $|w(z)| < 1$, for $z \in U$.

By giving specific values to A, B, λ, p and q, we obtain the following subclasses of λ-spiral functions studied by various authors in earlier works.

(i) Taking $q = 0$ and $p = 1$, the class $S_p^\lambda(A, B, q)$ coincides with the class $S^\lambda(A, B)$ studied by Dashrath and Shukla [3].

(ii) Taking $q = 0, \lambda = 0, A = (2\alpha\beta/p) - 1$ and $B = 2\beta - 1$, the class $S_p^\lambda(A, B, q)$ coincides with the class $S_p^*(\alpha, \beta)$ studied by Aouf [1].

(iii) Taking $q = 0$, $A = 1 - (2\alpha/p), B = -1$, the class $S_p^\lambda(A, B, q)$ coincides with the class $S^\lambda(p, \alpha)$ introduced by Patil and Thakare [6].

Let $T_p^\lambda(A, B, q)$ be the class of functions $g(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_n z^{n-p+1}$ analytic in $U' = \{z : 0 < |z| < 1\}$ and satisfying the condition

$$-e^{i\lambda}\frac{zg'(z)}{g(z)} < \frac{p + \{pB + (A-B)(p-q)\}z}{1 + Bz}\cos\lambda + ip\sin\lambda, z \in U',$$

where $-1 \leq B < A \leq 1$ and $\lambda \in (-\pi/2, \pi/2)$, $0 \leq q < p$, it follows from the definition of subordination that

$$-e^{i\lambda}\frac{zg'(z)}{g(z)} = \frac{p + \{pB + (A-B)(p-q)\}w(z)}{1 + Bw(z)}\cos\lambda + ip\sin\lambda, z \in U', \quad (1.2)$$

where $w(z)$ is analytic in U and satisfying the condition $w(0) = 0$, $|w(z)| < 1$, for $z \in U$.

Clearly for $p = 1$ and $q = 0$ we have the class $T^\lambda(A, B)$ considered by Dashrath and Shukla [3].

The purpose of this paper is to obtain sharp coefficient estimates for the classes $S_p^\lambda(A, B, q)$ and $T_p^\lambda(A, B, q)$ by using the method of Clunie [2], and maximization of $|a_{p+2} - \mu a_{p+1}^2|$ over the class $S_p^\lambda(A, B, q)$ for a given real as-well-as a complex number μ.

It is worthwhile to mention that some known results appear to be particular cases of our results.

2. Lemmas

The following lemma is to be found in Nehari [5, p.172].

Lemma 2.1. *If $w(z)$ is analytic in U and satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$, then $|w(z)| \leq |z|$ and that if $w(z) = \sum_{k=1}^{\infty} b_k z^k$, then*

$$|b_1| \leq 1$$

and

$$|b_2| \leq 1 - |b_1|^2. \tag{2.1}$$

The following lemma is due to Keogh and Merkes [4], the proof of which may be given by using Lemma 2.1.

Lemma 2.2. *Let $w(z) = \sum_{k=1}^{\infty} b_k z^k$ be analytic with $|w(z)| < 1$ in U. If S is any complex number, then*

$$|b_2 - Sb_1^2| \leq \max(1, |S|). \tag{2.2}$$

Equality may be attained with the functions $w(z) = z^2$ and $w(z) = z$.

Lemma 2.3. *If m is natural number such that $m \geq 2$, then*

$$\begin{aligned} &\frac{\cos^2\lambda}{m^2}\Big[(A-B)^2(p-q)^2 + \sum_{k=1}^{m-1}\big(\{(A-B)(p-q) - Bk\}^2 \\ &\quad -k^2\{1+(1-B^2)\tan^2\lambda\}\big) \times \prod_{j=0}^{k-1} u_j\Big] \\ &= \prod_{j=0}^{m-1} u_j \end{aligned} \tag{2.3}$$

where

$$u_j = \frac{|(A-B)(p-q)\cos\lambda e^{-i\lambda} - Bj|^2}{(j+1)^2}, \quad \textit{for } j = 0, 1, 2, 3, \cdots \tag{2.4}$$

Proof. We prove the lemma by induction on m. For $m = 2$ lemma is obvious. Next suppose that the result is true for $m = \ell - 1$, then for $m = \ell$ the left member of (2.3) reduces to

$$\frac{\cos^2\lambda}{\ell^2}\{(A-B)^2(p-q)^2 + \sum_{k=1}^{\ell-2}[\{(A-B)(p-q)-Bk\}^2$$
$$-k^2\{1+(1-B^2)\tan^2\lambda\}]\prod_{j=0}^{k-1}u_j$$
$$+[\{(A-B)(p-q)-B(\ell-1)\} - (\ell-1)^2\{1+(1-B^2)\tan^2\lambda\}]\prod_{j=0}^{\ell-2}u_j\}$$
$$= \frac{1}{\ell^2}\{(\ell-1)^2\prod_{j=0}^{\ell-2}u_j + \cos^2\lambda[\{(A-B)(p-q)-B(\ell-1)\}^2$$
$$-(\ell-1)^2\{1+(1-B^2)\tan^2\lambda\}]\prod_{j=0}^{\ell-2}u_j\}$$
$$= \frac{1}{\ell^2}[\{(A-B)(p-q)-B(\ell-1)\}^2\cos^2\lambda + B^2(\ell-1)^2\sin^2\lambda]\prod_{j=0}^{\ell-2}u_j$$
$$= \prod_{j=0}^{\ell-1}u_j,$$

showing that (2.3) is valid for $m = \ell$, and we are done.

3. Main Results

Theorem 3.1. *If* $f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p}z^{n+p} \in S_p^{\lambda}(A,B,q)$, *then*

$$|a_{p+1}| \le (A-B)(p-q)\cos\lambda; \tag{3.1}$$

$$|a_{n+p}| \le \frac{(A-B)(p-q)\cos\lambda}{n}, n \ge 2, \tag{3.2}$$

for $\{(A-B)(p-q)-B\} \le \sqrt{\{1+(1-B^2)\tan^2\lambda\}}$; and

$$|a_{n+p}| \le \prod_{j=0}^{n-1}u_j^{\frac{1}{2}}, n \ge 2 \tag{3.3}$$

for $\{(A-B)(p-q)-(n-1)B\} > (n-1)\sqrt{\{1+(1-B^2)\tan^2\lambda\}}$, where u_j is defined by (2.4) for $j = 0,1,2,3,\cdots$. The bounds (3.1), (3.2) and (3.3) are sharp.

Proof. By (1.1) we have

$$\begin{aligned}&e^{i\lambda}\sec\lambda z f'(z) - p(1+i\tan\lambda)f(z)\\&=[\{Bp+(A-B)(p-q)+iB\tan\lambda\}f(z) - Be^{i\lambda}\sec\lambda f'(z)]w(z)\end{aligned}$$

that is,

$$\begin{aligned}&(1+i\tan\lambda)\sum_{k=1}^{\infty} ka_{k+p}z^{k+p}\\&=[\sum_{k=0}^{\infty}\{Bp+(A-B)(p-q)+iB\tan\lambda - B(k+p)e^{i\lambda}\sec\lambda\}a_{k+p}z^{k+p}]w(z),\end{aligned}$$

where $a_p = 1$. Since $w(z)=\sum_{k=1}^{\infty} b_k z^k$ and $d_{k+p}=\{(1+i\tan\lambda)ka_{k+p}-c_k\}$ we obtain for $n\geq 2$,

$$\begin{aligned}&(1+i\tan\lambda)\sum_{k=1}^{n} ka_{k+p}z^{k+p}+\sum_{k=n+1}^{\infty} d_{k+p}z^{k+p} = [\sum_{k=0}^{n-1}\{Bp+(A-B)(p-q)\\&+iB\tan\lambda - B(k+p)e^{i\lambda}\sec\lambda\}a_{k+p}z^{k+p}]w(z), \qquad (3.4)\end{aligned}$$

where $\sum_{k=n+1}^{\infty} d_{k+p}z^{k+p}$ converges in U. Since (3.4) has the form $F(z) = G(z)\ w(z)$, where $|w(z)|<1$, it follows that

$$\frac{1}{2\pi}\int_0^{2\pi}|F(re^{i\theta})|^2 d\theta \leq \frac{1}{2\pi}\int_0^{2\pi}|G(re^{i\theta})|^2 d\theta \text{ for } 0<r<1. \qquad (3.5)$$

By substituting the values of $F(z)$ and $G(z)$ in (3.5) we have

$$\begin{aligned}&\sec^2\lambda\sum_{k=1}^{n} k^2|a_{k+p}|^2 r^{2(k+p)}+\sum_{k=n+1}^{\infty}|d_{k+p}|^2 r^{2(k+p)}\\&\leq\sum_{k=0}^{n-1}[\{(A-B)(p-q)-Bk\}^2+B^2k^2\tan^2\lambda]|a_{k+p}|^2 r^{2(k+p)}. \qquad (3.6)\end{aligned}$$

By letting $r\to 1$ in (3.6) we conclude that

$$\sec^2\lambda\sum_{k=1}^{n} k^2(|a_{k+p}|^2 \leq \sum_{k=0}^{n-1}[\{(A-B)(p-q)-Bk\}^2+B^2k^2\tan^2\lambda]|a_{k+p}|^2$$

which may be written as

$$\begin{aligned}|a_{n+p}| \ \leq\ & \frac{\cos^2\lambda}{n^2}[(A-B)^2(p-q)^2+\sum_{k=1}^{n-1}(\{(A-B)(p-q)-Bk\}^2\\&-k^2\{1+(1-B^2)\tan^2\lambda\})|a_{k+p}|^2] \text{ for } n=1,2,3,\cdots. \qquad (3.7)\end{aligned}$$

Inequality (3.1) follows from (3.7).

Further, $\{(A-B)(p-q)-B\} \leq \sqrt{\{1+(1-B^2)\tan^2\lambda\}}$ implies that $\{(A-B)(p-q)-(n-1)B\} \leq (n-1)\sqrt{\{1+(1-B^2)\tan^2\lambda\}}$, $n \geq 2$, and all the terms under the summation in (3.7) are non-positive and hence we conclude that

$$|a_{n+p}| \leq \frac{(A-B)(p-q)\cos\lambda}{n}, \{(A-B)(p-q)-B\} \leq \sqrt{\{1+(1-B^2)\tan^2\lambda\}}, n \geq 2.$$

The equality in (3.1) and (3.2) is attained for the function.

$$f(z) = \begin{cases} z^p(1+Bz^{n-1})^{\frac{(A-B)(p-q)\cos\lambda e^{-i\lambda}}{B(n-1)}}, & B \neq 0 \\ z^p \exp\{\frac{[Ap-q(A-B)]z^{n-1}\cos\lambda e^{-i\lambda}}{(n-1)}\}, & B = 0. \end{cases}$$

Now we prove (3.3) when $\{(A-B)(p-q)-(n-1)B\} > (n-1)\sqrt{\{1+(1-B^2)\tan^2\lambda\}}$, $n \geq 2$. All the terms under the summation are positive. We prove the result by induction on n. Suppose (3.3) holds for $n = m-1$ where $m \geq 2$. Then for $n = m$ we obtain from (3.7)

$$\begin{aligned} |a_{m+p}|^2 &\leq \frac{\cos^2\lambda}{m^2}[(A-B)^2(p-q)^2 + \sum_{k=1}^{m-1}(\{(A-B)(p-q)-Bk\}^2 \\ &\quad -k^2\{1+(1-B^2)\tan^2\lambda\})|a_{k+p}|^2] \\ &\leq \frac{\cos^2\lambda}{m^2}[(A-B)^2(p-q)^2 + \sum_{k=1}^{m-1}(\{(A-B)(p-q)-Bk\}^2 \\ &\quad -k^2\{1+(1-B^2)\tan^2\lambda\})\prod_{j=0}^{k-1}U_j] \\ &= \prod_{j=0}^{m-1}U_j, \text{ by lemma (2.3).} \end{aligned}$$

So (3.3) holds for all $n \geq 2$, and hence

$$|a_{n+p}| \leq \prod_{j=0}^{n-1}U_j^{\frac{1}{2}}.$$

The equality in (3.3) is attained for the function

$$f(z) = \begin{cases} z^p(1+Bz)^{\frac{(A-B)(p-q)\cos\lambda e^{-i\lambda}}{B}}, & B \neq 0 \\ z^p\exp[\{Ap-q(A-B)\}z\cos\lambda e^{-i\lambda}], & B = 0. \end{cases}$$

This completes the proof of the theorem.

Remark. (1) Putting $q = 0$, $p = 1$ in Theorem (3.1), we get the result obtained by Dashrath and Shukla [3].

(2) Putting $q = 0, \lambda = 0, A = (2\alpha\beta/p) - 1, B = 2\beta - 1$ in Theorem (3.1), we get the result obtained by Aouf [1].

(3) Putting $q = 0, A = 1 - (2\alpha/p)$ and $B = -1$ in Theorem (3.1), we get the result obtained by Patil and Thakare [6].

Theorem 3.2. *If* $f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \in S_p^\lambda(A, B, q)$, *then*

(a) *for any real number* μ, *we have*

$$|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} \frac{(A-B)(p-q)\cos\lambda}{2}[\cos\lambda\{(A-B)(p-q)(1-2\mu) - B\} + |B\sin\lambda|] \\ \quad \text{if } \mu \leq \frac{(A-B)(p-q)-(B+1)}{2(A-B)(p-q)}, \\ \frac{(A-B)(p-q)\cos\lambda}{2}[\cos\lambda + |B\sin\lambda|] \\ \quad \text{if } \frac{(A-B)(p-q)-(B+1)}{2(p-q)(A-B)} \leq \mu \leq \frac{(A-B)(p-q)+(1-B)}{2(A-B)(p-q)}, \\ \frac{(A-B)(p-q)\cos\lambda}{2}[\cos\lambda\{(A-B)(p-q)(2\mu-1) + B\} + |B\sin\lambda|] \\ \quad \text{if } \mu \geq \frac{(p-q)(A-B)+(1-B)}{2(A-B)(p-q)}, \end{cases} \tag{3.8}$$

(b) *for any complex number* μ, *we have*

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{(p-q)(A-B)\cos\lambda}{2}\max\{1, |(p-q)(A-B)(2\mu-1)\cos\lambda + Be^{i\lambda}|\} \tag{3.9}$$

The result is sharp for each μ *either real or complex.*

Proof. From (1.1) we have

$$\frac{p + \{Bp + (A-B)(p-q)\}w(z)}{1 + Bw(z)} = e^{i\lambda}\sec\lambda\frac{zf'(z)}{f(z)} - ip\tan\lambda, \tag{3.10}$$

where $w(z) = \sum_{k=1}^{\infty} b_k z^k, w(0) = 0$ and $|w(z)| < 1$ for $z \in U$. From (3.10) we obtain

$$\begin{aligned} w(z) &= \frac{p(1 + i\tan\lambda)f(z) - e^{i\lambda}\sec\lambda zf'(z)}{Be^{i\lambda}\sec\lambda zf'(z) - \{Bp + (A-B)(p-q) + iBp\tan\lambda\}f(z)} \\ &= \frac{(1 + i\tan\lambda)\sum_{k=1}^{\infty} ka_{k+p}z^k}{(A-B)(p-q) + \sum_{k=1}^{\infty}[(A-B)(p-q) - Bk - iBk\tan\lambda]a_{k+p}z^k} \\ &= \frac{(1 + i\tan\lambda}{(A-B)(p-q)}[\{a_{p+1}\}z + \{2a_{p+2} \\ &\quad - \frac{(A-B)(p-q) - B - iB\tan\lambda}{(A-B)(p-q)}a_{p+1}^2\}z^2 + \cdots]. \end{aligned}$$

Comparing the coefficients of z and z^2 on both sides, we have

$$\begin{aligned} b_1 &= \frac{(1+i\tan\lambda)}{(A-B)(p-q)}a_{p+1}, \\ b_2 &= \frac{(1+i\tan\lambda)}{(A-B)(p-q)}\{2a_{p+2} \\ &\quad -\frac{(A-B)(p-q)-B-iB\tan\lambda}{(A-B)(p-q)}\cdot a_{p+1}^2\}. \end{aligned}$$

Thus

$$\begin{aligned} a_{p+1} &= \frac{(A-B)(p-q)}{e^{i\lambda}\sec\lambda}b_1 \\ a_{p+2} &= \frac{(A-B)(p-q)}{2e^{i\lambda}\sec\lambda}b_2 \\ &\quad +\frac{(A-B)(p-q)-B-iB\tan\lambda}{2(A-B)(p-q)}\cdot a_{p+1}^2. \end{aligned}$$

Hence

$$\begin{aligned} |a_{p+2}-\mu a_{p+1}^2| &= \frac{(A-B)(p-q)\cos\lambda}{2}|b_2-\{(A-B)(p-q)(2\mu-1) \\ &\quad +Be^{i\lambda}\sec\lambda\}\cdot\frac{b_1^2}{e^{i\lambda}\sec\lambda}|. \end{aligned} \tag{3.11}$$

(a) When μ is real, (3.11) becomes

$$\begin{aligned} |a_{p+2}-\mu a_{p+1}^2| &\leq \frac{(A-B)(p-q)\cos\lambda}{2}[|b_2|+|(A-B)(p-q)(2\mu-1)\cos\lambda \\ &\quad +Be^{i\lambda}||b_1|^2] \end{aligned} \tag{3.12}$$

$$\begin{aligned} |a_{p+2}-\mu a_{p+1}^2| &\leq \frac{(A-B)(p-q)\cos\lambda}{2}[1+\{|(A-B)(p-q)(2\mu-1)\cos\lambda \\ &\quad +Be^{i\lambda}|-1\}|b_1|^2]. \end{aligned} \tag{3.13}$$

Again using Lemma 2.1 for $|b_1|$ in (3.13) we are led to

$$\begin{aligned} |a_{p+2}-\mu a_{p+1}^2| &\leq \frac{(A-B)(p-q)\cos\lambda}{2}[|(A-B)(p-q)(2\mu-1) \\ &\quad +B|\cos\lambda+|B\sin\lambda|]. \end{aligned} \tag{3.14}$$

Thus from (3.14) we can simply obtain the result of Theorem 3.2 as stated in (a) for various values of real μ.

(b) When μ is a complex number, (3.11) may be written as

$$|a_{p+2} - \mu a_{p+1}^2| = \frac{(A-B)(p-q)\cos\lambda}{2}|b_2 - \{\frac{(A-B)(p-q)(2\mu-1)\cos\lambda + Be^{i\lambda}|}{e^{i\lambda}}\}b_1^2|. \quad (3.15)$$

Using lemma 2.2 in (3.15) we obtain

$$|a_{p+2} - \mu a_{p+1}^2| \le \frac{|(A-B)(p-q)\cos\lambda}{2}\max\{1, |(A-B)(p-q)(2\mu-1)\cos\lambda + Be^{i\lambda}|\}$$

which is (3.9) in (b) of Theorem 3.2.

Remarks 1. Putting $q=0$, $A = 1-(2\alpha/p)$ and $B=-1$ in Theorem 3.2, we get the result obtained by Patil and Thakare [6].

2. Putting $q=0, p=1$ in Theorem 3.2, we get the result obtained by Dashrath and Shukla [3].

Theorem 3.3. *If $g(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_n z^{n-p+1} \in T_p^{\lambda}(A,B,q)$ then $|b_n| \le \frac{(A-B)(p-q)\cos\lambda}{(n+1)}$, for $n = 0,1,2,3,\cdots$. The result is sharp.*

Proof. The proof is based on the steps of the theorem (3.1). The following functions give sharp estimate

$$g(z) = \begin{cases} \frac{1}{z^p}(1+Bz^{n+1})^{\frac{(B-A)(p-q)\cos\lambda e^{-i\lambda}}{B(n+1)}}, & B \neq 0 \\ \frac{1}{z^p}\exp[\frac{\{(B-A)(p-q)-Bp\}z^{n+1}\cos\lambda e^{-i\lambda}}{n+1}], & B = 0. \end{cases}$$

Remark. Putting $q=0$, $p=1$ in Theorem 3.3, we get the result obtained by Dashrath and Shukla [3].

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