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SEMIGROUPS AND HIGHER ORDER GENERATORS

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The semigroup theory, the properties of its generators and its applieca-tions to the mixed problems and the initial value problems of partial differential equations have been considered by several authors (see for examples [1], [2], [4] and [6]). Here we will show that if the operator A generates a contraction semigroup, then under certain conditions (which will be stated) the operators A^{2m}, A^m and $\sum_{k=1}^{m} A^k$ generate analytic semigroups. Some examples of generators will be given to illustrate the results, and higher order equations of evolution will be considered.

1. Introduction

Let X be a Hilbert space and A be a linear operator defined in X. Let us consider the equation of evolution

$$\frac{du(t)}{dt} + Au(t) = f(t). \tag{1.1}$$

It is known [8] that if the operator A is closed densely in X, then A generates a contraction semigroup $\{T(t), t \geq 0\}$ of bounded linear operators if and only if A and its adjoint A^* are maximal dissipative. Now some definitions are listed for later use.

Definition 1.1. Let A be linear operator in the Hilbert space X, and its domain is assumed to be dense in X. The operator A is said to be dissipative if $Re(Au, u) \leq 0$ or equivalently

$$\|(A-\lambda)u\| \geq Re\lambda\|u\|, \text{ for all } u \in D(A) \text{ and } Re\lambda > 0.$$

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If $Re(Au, u) \geq 0$, that is $-A$ is dissipative, A is said to be accretive. A dissipative operator which extends a dissipative operator is called a dissipative extension of A. An operator A is said to be maximal dissipative if its only dissipative extension is the operator A itself. Accerative extensions and maximal accretive operators are defined similarly.

Definition 2.1. Let $T(t)$ for each $t \in [0, \infty)$ be a bounded linear operator in X, $\{T(t)\}$ is called a semigroup of bounded operators if

(1) $T(t+s) = T(t) \quad T(s) = T(s)T(t), \quad s, t \geq 0$

(2) $T(0) = I$

(3) $T(t)$ is strongly continuous in $t \in [0, \infty)$.

It is known [8, p.53] that there exists real numbers $M \geq 1$ and β such that

$$\|T(t)\| \leq Me^{\beta t} \tag{1.2}$$

for all $t \geq 0$.

A semigroup $\{T(t)\}$ is called a contraction semigroup if it satisfies

$$\|T(t)\| \leq 1.$$

When A generates a semigroup satisfies (1.2), we write $A \in G(X, M, \beta)$, and when the semigroup is a contraction semigroup we write $A \in G(X, 1, 0)$.

2. Higher Order Generators

Here we assume that $A \in G(X, 1, 0)$. Using the results of [3], [5] and [8] we have the following lemmas.

Lemma 2.1. *If $A \in G(X, 1, 0)$, then $D(A^{2m})$ is dense in X, $m = 1, 2, 3, \cdots$*

Proof. Since $A \in G(X, 1, 0)$, it follows that A is closed densely defined in X, and A and A^* are maximal dissipative. Hence $-A$ is maximal accretive and therefore it is of type $(\frac{\pi}{2}, 1)$ (for notations see [8] p.32) from which we can deduce [8] that

$$D((-A)^{2m}) = D(A^{2m}) \text{ is dense in } X, \text{ where } m = 1, 2, 3, \cdots$$

Lemma 2.2. *Let $A \in G(X, 1, 0)$. If $A^*(t) = A(t)$ or $A^*(t) = -A(t)$, then A^{2m} is closed.*

Proof. Since $(A^{2m})^* = (A^*)^{2m} = (\pm A)^{2m} = A^{2m}$ it follows that A^{2m} is self-adjoint. Now A^{2m} is self-adjoint and densely (from Lemma (2.1)) defined in X, hence [3] it is closed operator in X.

Theorem 2.1. *Let $A \in G(X,1,0)$, if $A^* = -A$ then*

$$(-1)^{1+m}A^{2m} \in G(X,1,0)$$

Proof. From Lemmas (2.1) and (2.2), the operator A^{2m} is closed densely defined in X. Now let m be an odd positive integer, we have

$$\begin{aligned} Re((A^{2m}-\gamma)u,u) &= Re(A^m u, A^{*m}u) - Re\gamma\|u\|^2 \\ &= Re(A^m u, -A^m u) - Re\|u\|^2 \\ &= -\|A^m u\|^2 - Re\gamma\|u\|^2 \\ &\leq -Re\|u\|^2, \end{aligned}$$

and so

$$\|(A^{2m}-\gamma)u\|\|u\| \geq -Re((A^{2m}-\gamma) \geq -Re((A^{2m}-\gamma)u,u) \geq Re\gamma\|u\|^2$$

i.e.

$$\|(A^{2m}-\gamma)u\| \geq Re\gamma\|u\|, Re\gamma > 0 \tag{2.1}$$

therefore A^{2m} and its adjoint are dissipative, from which we can deduce [8] that they are maximal dissipative in X, and then

$$A^{2m} \in G(X,1,0) \tag{2.2}$$

Secondly let m be an even positive integer, we have

$$\begin{aligned} Re(A^{2m}+\gamma)u,u) &= Re(A^m u, A^{*m}u) + Re\gamma\|u\|^2 \\ &= Re(A^m u, A^m u) + Re\gamma\|u\|^2 \\ &= \|A^m u\|^2 + Re\gamma\|u\|^2 \\ &\geq Re\gamma\|u\|^2, Re\gamma > 0 \end{aligned}$$

and so

$$\|(A^{2m}+\gamma)u\| < Re\gamma\|u\|, \tag{2.3}$$

therefore $-A^{2m}$ and its adjoint are dissipative, from which we can deduce [8] that they are maximal dissipative, and then

$$-A^{2m} \in G(X,1,0). \tag{2.4}$$

Combining (2.2) and (2.4) we get

$$(-1)^{1+m}A^{2m} \in G(X,1,0), \tag{2.5}$$

where $m = 1, 2, \cdots$

Theorem 2.2. *Let $A \in G(X,1,0)$. If A is self-adjoint, then*

$$-A^{2m} \in G(X,1,0), \text{ and } m = 1,2,\cdots$$

Proof. From Lemmas (2.1) and (2.2), A^{2m} is closed densely defined in X. Now since

$$\begin{aligned} Re((A^{2m}+\gamma)u,u) &= Re(A^m u, A^{*m}u) + Re\gamma\|u\|^2 \\ &= Re(A^m u, A^m u) + Re\gamma\|u\|^2 \\ &\geq Re\gamma\|u\|^2, Re\gamma > 0 \end{aligned}$$

and so

$$\|(A^{2m}+\gamma)u\| \geq Re\gamma\|u\| \tag{2.6}$$

It follows that $-A^{2m}$ and its adjoint are dissipative and they are maximal dissipative in X (cf. [8]). Hence

$$-A^{2m} \in G(X,1,0), \text{ where } m = 1,2,3\cdots$$

which completes the prove.

Theorem 2.3. *Let $A \in G(X,1,0)$ and $A^* = A$. If A has a bounded inverse, then*

$$-A^m \in G(X,1,0), \quad m = 1,2,3\cdots.$$

Proof. Since A^m is self-adjoint and has a bounded inverse, it follows that

$$(A^{2m}u,u) = (A^m u, A^m u) = \|A^m u\|^2 > 0$$

for nonvanishing $u \in D(A^m)$. So A^{2m} is strictly positive operator, hence A^{2m} has a strictly positive square root A^m (cf.[7]) i.e.

$$(A^m u, u) > 0 \text{ for } u \in D(A^m). \tag{2.7}$$

Now since A^m is the square root of A^{2m}, it follows from Lemmas (2.1) and (2.2) (and Proposition 2.3.1 of [8]) that A^m is closed densely defined in

X, with domain $D = D(A^m)$. From (2.7) A^m and its adjoint are accretive in X, consequently $-A^m$ and its adjoint are dissipative. Hence they are maximal dissipative and therefore

$$-A^m \in G(X, 1, 0).$$

Corollary 2.1. *If the assumptions of Theorem (2.3) are satisfied, then*

$$L = -\sum_{k=1}^{m} A^k \in G(X, 1, 0).$$

Proof. From Theorem (2.3), $-A^m \in G(X, 1, 0)$, so it is of type $(\frac{\pi}{2}, 1)$ (cf [8]). Also A^m has a bounded inverse, so (by Proposition 2.3.1 of [8]).

$$D(a^\alpha) \supset D(A^\beta), \text{ for } 0 < \alpha < \beta.$$

Now for the operator L, take $D(L) = D(A^m)$, and since

$$(Lu, u) = -\sum_{k=1}^{m} (A^k u, u) \leq 0$$

it follows that L and L^* are dissipative, consequently they are maximal dissipative and

$$L \in G(X, 1, 0).$$

Corollary 2.2. *If $B \in G(X, 1, 0)$, then the semigroup generated by B is analytic. So the semigroups(contraction) in Theorems (2.1), (2.2) and (2.3) and Corollary (2.1) are analytic.*

Proof. If $B \in G(X, 1, 0)$, then $-B$ and its adjoint are maximal accretive, from which we deduce [8] that $-B$ is of type $(\frac{\pi}{2}, 1)$. So (Theorem 3.3.1 of [8]) the operator B generates an analytic semigroup. From the uniqueness of the semigroup generated by B, we get the result.

3. Applications

Consider the equation of evolution

$$\frac{du(t)}{dt} - \sum_{k=1}^{m} A^k u(t) = f(t) \tag{3.1}$$

with the initial data

$$u(0) = u_0. \tag{3.2}$$

From our results here and as in the case of first order generators, we can easely prove the following theorem.

Theorem 3.1. *Let A satisfy the assumptions of Theorem (2.3). If $u_0 \in X$ and $f(t) \in C(X, I)$, then the initial value problem (3.1) and (3.2) has a unique solution $u(t) \in C(X, I)$, given by*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds, \tag{3.3}$$

with the following properties (for $t > 0$).

(1) $\frac{du(t)}{dt} \in C(X, I)$, *and*

$$\|\frac{du(t)}{dt}\| \leq \frac{c}{t}\|u_0\| + \|f(t)\| + \frac{c}{t}\int_0^t \|f(s)\|ds \tag{3.4}$$

(2) $u(t) \in D(A^m)$, *and*

$$\|\sum_{k=1}^{m} A^k u(t)\| \leq \frac{c}{t}\|u_0\| + c\int_0^t \|f(s)\|ds \tag{3.5}$$

where $\{T(t)\}$ is the semigroup (analytic) generated by $-\sum_{k=1}^{m} A^k$, and c is a positive constant.

Examples of generators

(1) Let ω be a bounded open subset of R_n, with boundary $\partial\omega$, and define the operator A as $D(A) = \{u \in L_2(\omega) : \Delta u \in L_2(\omega), u|_{\partial\omega} = 0\}$ and $Au = -\Delta u$. It is proved in [1] that $A \in G(X, 1, 0)$ and the operator A has a bounded inverse. So it satisfies the assumptions of Theorem (2.3). Then the operator $-\sum_{k=1}^{m}(-\Delta)^k$ generates an analytic semigroup. Now consider the mixed problem

$$\frac{\partial u(x,t)}{\partial t} - \sum_{k=1}^{m}(-\Delta)^k u(x,t) = f(x,t), x \in \omega, t > 0 \tag{3.6}$$

$$u(x,t) = 0, x \in \partial\omega, t > 0 \tag{3.7}$$

$$u(x,0) = u_0(x), x \in \omega \tag{3.8}$$

where $u_0(x) \in L_2(\omega)$ and $f(x,t) \in C(L_2(\omega), I)$. Applying Theorem (3.1) to the mixed problem (3.6), (3.7) and (3.8), we deduce that this mixed problem has a unique solution

$$u(x,t) \in W_2^{2m}(\omega)$$

and continuous with respect to $t \in I$. Also this solution satisfies (3.4) and (3.5). Now from Sobolev's embedding Theorem [7] we have the following corollary.

Corollary 3.1. *If $2m > \frac{n}{2} + k$, then the solution of the mixed problem (3.6), (3.7) and (3.8)*

$$u(x,t) \in C^k(\omega),$$

and $\frac{\partial u(x,t)}{\partial t}$ exists in the usual sence.

(2) *Let $X = L_p(R_n)$, $1 < p < x$, put $G(t,x) = (2\sqrt{\pi t})^{-n} \exp(-|x|^2/4t)$, for each $t > 0$ and each $x \in R_n$. The contraction semigroup $\{T(t)\}$ defined by*

$$T(t)u(x) = \int_{R_n} G(t, x-y)u(y)dy$$

has the generator given by $Au = -\Delta u$, with domain $D(A) = W_p^2(R_n)$, $1 < p < \infty$, and $A^* = A$. So this operator satisfies the assumptions of Theorem (2.2), therefore $-A^{2m} = -\Delta^{2m}$ generates an analytic semigroup.

(3) *Let the matrices $a_j(t)$, for $j = 1, 2, \cdots, n$ and $t > 0$, be Hermitian. For each $u = (u_1, u_2, \cdots u_N)^t \in L_2(R_n)^N = X$, define*

$$A(t)u = \sum_{j=1}^{n} a_j(t) \frac{\partial u}{\partial x_j} \tag{3.9}$$

with domain $D(A) = \{u \in X : A(t)u \in X\} C W_2^1(R_n)^N$. It is known [8] that for each $\beta > 0$ the operator $\pm A(t) \in G(X, 1, \beta)$ and $A^* = -A$. So this operator (3.9) satisfies the assumptions of theorem (2.1), therefore $(-1)^{m+1} A^{2m}(t)$ generates an analytic semigroup.

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