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A NOTE ON INTEGRAL INEQUALITIES OF THE WENDROFF TYPE

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This paper is devoted to some nonlinear inequalities of Wenderoff type. The bound obtained by these inequalities are adequate in many applications in the theory of partial differential and integral equations.

1. Introduction

The inequality of Gronwall [6] and its generalizations have played a very important role in the analysis of differential and integral equations. An interesting but apparently neglected generalization of Gronwall's inequalities in two independent variables is due to Wendroff given in [3, p. 154]. In recent papers of Snow [11], Agarwal [2], Corduneanu [4], Kasture and Deo [7], Pachpatte [8], Shastri and Kasture [10], Rasmussen [9], and Vaz and Deo [12], some nonlinear generalizations to Wendroff inequality are given. In this paper, we derive an analogous results which are extensions to those results in [3], [4], [5], [7], [9], & [10]. The bound obtained by the new inequalities are adequate in many applications in the theory of partial differential and integral equations.

2. Main Results

In this section we shall state and prove some partial integral inequalities in two independent variables.

Theorem 2.1. *Assume*

(i) $\phi(x,y), g(x,y)$ *and* $C(x,y)$ *are real-valued continuous functions defined on a domain* $D = [x_0, a] \times [y_0, b]$

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(ii) $q(x,y) \geq 1$, *and is a real valued continuous function defined on* D.

(iii) $f(x,y)$ *is a positive continuous and nondecreasing in both its arguments and defined on* D.

(iv) $K(x,y,s,t,\phi)$ *and* $W(x,y,\phi)$ *are real-valued nonnegative continuous functions defined on* $D^2 \times R$ *and* $D \times R$ *respectively (where* R *is the set of real numbers) and nondecreasing in the last variable and* $K(x,y,s,t,\phi)$ *is uniformly Lipschitz in the last variable. If the inequality*

$$\begin{aligned}\phi(x,y) \leq\ & f(x,y) + q(x,y)[\int_{x_0}^{x}\int_{y_0}^{y} g(s,t)\phi(s,t)dsdt \\ & + \int_{x_0}^{x}\int_{y_0}^{y} g(s,t)q(s,t)(\int_{x_0}^{s}\int_{y_0}^{t} C(\xi,\zeta)\phi(\xi,\zeta)d\xi d\zeta)dsdt] \\ & + W(w,y,\int_{x_0}^{x}\int_{y_0}^{y} K(x,y,s,t,\phi(s,t)dsdt) \qquad (2.1)\end{aligned}$$

is satisfied, then

$$\phi(x,y) \leq \psi(x,y)[f(x,y) + W(w,s,r(x,y)], \qquad (2.2)$$

for all $(x,y) \in D$, *where*

$$\begin{aligned}\psi =\ & q(x,y)\exp[\int_{x_0}^{x}\int_{y_0}^{y} g(s,t)q(s,t)\{1 + \int_{x_0}^{t}\int_{y_0}^{s} C(\xi,\zeta) \\ & \times q(\xi,\zeta))d\xi d\zeta\}dsdt], \qquad (2.3)\end{aligned}$$

and $r(x,y)$ *is the maximal solutions of*

$$r(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} K(x,y,s,t,\psi(s,t)[f(s,t)W(s,t,r(s,t))])dsdt \qquad (2.4)$$

existing on D.

Proof. Define

$$m(x,y) = f(x,y) + W(x,y)\int_{x_0}^{x}\int_{y_0}^{y} K(x,y,s,t,\phi(s,t))dsdt. \qquad (2.5)$$

Then we can write inequality (2.1) in the form

$$\begin{aligned}\phi(x,y) \leq\ & m(x,y) + q(x,y)[\int_{x_0}^{x}\int_{y_0}^{y} g(s,t)\phi(s,t)dsdt \\ & + \int_{x_0}^{x}\int_{y_0}^{y} g(s,t)q(s,t)\{\int_{x_0}^{x}\int_{y_0}^{y} C(\xi,\zeta)\phi(\xi,\zeta\}d\xi d\zeta)dsdt] \qquad (2.6)\end{aligned}$$

Since $q(x,y) \geq 1$, and $m(x,y)$ is positive and nodecreasing, so inequality (2.6) has the form

$$\begin{aligned} \frac{\phi(x,y)}{m(x,y)} &< q(x,y)[1 + \int_{x_0}^{x}\int_{y_0}^{y} g(s,t)\frac{\phi(s,t)}{m(s,t)}dsdt \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} g(s,t)q(s,t) \\ &\quad \times (\int_{x_0}^{x}\int_{y_0}^{y} C(\xi,\zeta)\frac{\phi(\xi,\zeta)}{m(\xi,\zeta)}d\xi d\zeta)dsdt]. \end{aligned} \tag{2.7}$$

Define a function $u(x,y)$ such that

$$\begin{aligned} u(x,y) &= 1 + \int_{x_0}^{x}\int_{y_0}^{y} g(s,t)\frac{\phi(s,t)}{m(s,t)}dsdt \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} g(s,t)q(s,t) \\ &\quad \times (\int_{x_0}^{x}\int_{y_0}^{y} C(\xi,\zeta)\frac{\phi(\xi,\zeta)}{m(\xi,\zeta)}d\xi d\zeta)dsdt, \end{aligned} \tag{2.8}$$

with $u(x_0,y) = u(x,y_0) = 1$. Thus

$$u_{xy}(xy) = g(x,y)\frac{\phi(x,y)}{m(x,y)} + g(x,y)q(x,y).$$

$$\int_{x_0}^{x}\int_{y_0}^{y} C(\xi,\zeta)\frac{\phi(\xi,\zeta)}{m(\xi,\zeta)}d\xi d\zeta.$$

Using (2.7) we obtain

$$\begin{aligned} u_{xy}(x,y) &\leq g(x,y)q(x,y)u(x,y) + g(x,y)q(x,y) \\ &\quad \times \int_{x_0}^{x}\int_{y_0}^{y} C(\xi,\zeta)u(\xi,\zeta)\cdot q(\xi,\zeta)d\xi d\zeta. \end{aligned}$$

Hence

$$\frac{u_{xy}(x,y)}{u(x,y)} \leq g(x,y)q(x,y)[1 + \int_{x_0}^{x}\int_{y_0}^{y} C(\xi,\zeta)q(\xi,\zeta)d\xi d\zeta]. \tag{2.9}$$

From (2.9) we observe that

$$\begin{aligned} \frac{u(x,y)u_{xy}(xy)}{u^2(x,y)} &\leq g(x,y)q(x,y)[1 + \int_{x_0}^{x}\int_{y_0}^{y} C(\xi,\zeta)q(\xi,\zeta)d\xi d\zeta] \\ &\quad + \frac{u_x(x,y)u_y(x,y)}{u^2(x,y)}, \end{aligned}$$

i.e.

$$\frac{\partial}{\partial y}\left(\frac{u_x(x,y)}{u(x,y)}\right) \leq g(x,y)q(x,y)[1+\int_{x_0}^{x}\int_{y_0}^{y} C(\xi,\zeta)q(\xi,\zeta)d\xi d\zeta]. \qquad (2.10)$$

By keeping x fixed in (2.10), set $y = t$ and integrate with respect to t from y_0 to y and then keeping y fixed, set $x = s$ and integrate with respect to s from x_0 to x we can easily have

$$\begin{aligned} u(x,y) \;\leq\; & \exp\,[\int_{x_0}^{x}\int_{y_0}^{y} g(s,t)q(s,t)[1+\int_0^t\int_0^s C(\xi,\zeta) \\ & \times q(\xi,\zeta)d\xi d\zeta]dsdt]. \end{aligned}$$

Substituting this bound on $u(x,y)$ in (2.7) we have

$$\phi(x,y) \leq m(x,y)\psi(x,y) \qquad (2.11)$$

where $\psi(x,y)$ is as defined in (2.3). By using (2.5) and (2.11) we get

$$\phi(x,y) \leq \psi(x,y)[f(x,y) + W(x,y,\int_{x_0}^{x}\int_{y_0}^{y} K(x,y,s,t,\phi(s,t)))dsdt].$$

This complete the proof.

Theorem 2.2. *Assume*

(i) $u(x,y), g(x,y)$ *and* $h(x,y)$ *are real-valued nonnegative continuous functions defined on :*

$$\Delta = \{(x,y) : 0 \leq x < \infty, \quad 0 \leq y < \infty\},$$

(ii) $f(x,y)$ *is a real-valued positive, continuous function defined on* Δ

(iii) $W(r)$ *is a real-valued positive, continuous, monotonic, non-decreasing, subadditive and submultiplicative function for* $r \geq 0$.

(iv) $H(r)$ *is a real-valued positive, continuous, monotonic nondecreasing function for* $r \geq 0$. *If*

$$u(x,y) \leq f(x,y) + g(x,y)H(\int_0^x\int_y^\infty h(s,t)W(u(s,t))dsdt). \qquad (2.12)$$

is satisfied for all $(x,y) \in \Delta$, *then for all* $(x,y) \in \Delta_1 \subset \Delta$

$$\begin{aligned} u(x,y) \;\leq\; & f(x,y) + g(x,y)H[G^{-1}\{G(\int_0^\infty\int_0^\infty h(s,t)W(f(s,t))dsdt) \\ & + \int_0^\infty\int_y^\infty h(s,t)W(g(s,t))dsdt\}]. \end{aligned} \qquad (2.13)$$

where G and G^{-1} are defined by

$$G(r) = \int_{r_0}^{r} \frac{ds}{W(H(s))}, \quad r > r_0 \geq 0, \tag{2.14}$$

G^{-1} is the inverse function of G, and

$$G(\int_0^\infty \int_0^\infty h(s,t)W(f(s,t))dsdt + \int_0^x \int_y^\infty h(s,t) \\ W(g(s,t))dsdt) \in Dom(G^{-1}), \tag{2.15}$$

for all $(x,y) \in \Delta_1$, where $\Delta_1 = \{(x_1,y_1) : 0 \leq x_1 \leq x,\ 0 \leq y \leq y_1 < \infty\}$.

Proof. Without loss of generality we may assume that $u(x,y) \geq f(x,y)$. Using the subadditivity of W and monotonicity of H, we have from (2.12), that

$$\begin{aligned} u(x,y) - f(x,y) \ &\leq\ g(x,y)H[\int_0^x \int_y^\infty h(s,t)W(u(s,t) - f(x,t))dsdt. \\ &\quad + \int_0^\infty \int_0^\infty h(s,t)W(f(s,t))dsdt]. \end{aligned} \tag{2.16}$$

Let $X(x,y) = u(x,y) - f(x,y)$ and define

$$\begin{aligned} v(x,y) \ &=\ \int_0^x \int_y^\infty h(s,t)W(X(s,t))dsdt \\ &\quad + \int_0^\infty \int_0^\infty h(s,t)W(f(s,t))dsdt, \end{aligned} \tag{2.17}$$

$$v(\infty,\infty) = \int_0^\infty \int_0^\infty h(s,t)W(f(s,t))dsdt.$$

Then equation (2.16) can be restated as

$$X(x,y) \leq g(x,y)H(v(x,y)). \tag{2.18}$$

Differentiating (2.16) and using the monotonicity and submultiplicity of W, we get

$$\begin{aligned} v_{xy}(x,y) \ &\leq\ h(x,y)W(X(x,y)) \\ &\leq\ h(x,y)W(g(x,y))H(v(x,y)) \\ &\leq\ h(x,y)W(g(x,y))W(H(v(x,y))). \end{aligned} \tag{2.19}$$

From (2.18) we abserve that

$$\frac{W(H(v(x,y)))v_{xy}(x,y)}{W^2(H(v(x,y)))} \le h(x,y)W(g(x,y)) + \frac{v_x(x,y)W_y(H(v(x,y)))}{w^2(H(v(x,y)))},$$

i.e.

$$\frac{\partial}{\partial y}\Big(\frac{v_x(x,y)}{W(H(v(x,y)))}\Big) \le h(x,y)W(g(x,y)).$$

By keeping x fixed in the above inequality, we set $y = t$ and then integrating with respect to t from y to ∞, we have

$$-\frac{v_x(x,y)}{W(H(v(x,y)))} \le \int_y^\infty h(x,y)W(g(x,t))dt. \tag{2.20}$$

From (2.14) and (2.20) we observe that

$$-\frac{\partial}{\partial x}G(v(x,y)) \le \int_y^\infty h(x,t)W(g(x,t))dt.$$

By keeping y fixed in the above inequality, setting $x = s$ and then integrating with respect to s from 0 to ∞, we have:

$$\begin{aligned} v(x,y) \;\le\; & G^{-1}[G\int_0^\infty \int_0^\infty h(s,t)W(f(s,t))dsdt \\ & + \int_0^\infty \int_y^\infty h(s,t)W(g(s,t))dsdt], \end{aligned} \tag{2.21}$$

for $(x,y) \in \Delta_3$. The desired bound in (2.13) follows from (2.12), (2.18), (2.21).

Theorem 2.3. *Assume:*

(i) $u(x,y), a(x,y), g(x,y)$ *and* $h(x,y)$ *are real-valued nonnegative continuous functions defined on*

$$\Delta = \{(x,y) : 0 \le x < \infty, \quad 0 \le y < \infty\}.$$

(ii) $f(x,y)$ *is a real-valued positive, continuous function defined on* Δ.

(iii) $W(r)$ *is a real-valued positive, continuous, nondecreasing subadditive and submultiplicative function for* $r \ge 0$,

(iv) $H(r)$ *is a real-valued positive, continuous monotonic, non-decreasing function for* $r \geq 0$. *If*

$$\begin{aligned} u(x,y) \leq\ & f(x,y) + \int_x^\infty a(s,y)u(s,y)ds \\ & + g(x,y)H\left(\int_0^x \int_y^\infty h(s,t)W(u(s,t))dsdt\right), \end{aligned} \tag{2.22}$$

is satisfied for all $(x,y) \in \Delta$, *then for all* $(x,y) \in \Delta_2 \subset \Delta$

$$\begin{aligned} u(x,y) \leq\ & \phi(x,y)[f(x,y) + g(x,y)H(G^{-1}((G\int_x^\infty \int_y^\infty h(s,t) \\ & \times W(\phi(s,t)f(s,t))dsdt) + \int_0^x \int_y^\infty h(s,t)W(\phi(s,t)) \\ & \times g(s,t))]dsdt))]. \end{aligned} \tag{2.23}$$

where

$$\phi(x,y) = \exp\left(\int_x^\infty a(s,y)ds\right), \tag{2.24}$$

G *and* G^{-1} *as defined in Theorem 2.2 such that*

$$\begin{aligned} & G\left(\int_x^\infty \int_y^\infty h(s,t)W(\phi(s,t)f(s,t)dsdt\right) \\ & \int_0^x \int_y^\infty h(s,t)W(\phi(s,t)g(s,t)dsdt \in Dom(G^{-1}), \end{aligned}$$

for all $(x,y) \in \Delta_2$, *where* $\Delta_2 = \{(x_2,y_2) : 0 \leq x_2 \leq x,\ 0 \leq y \leq y_2 < \infty\}$.

Proof. Define a function $m(x,y)$ by

$$m(x,y) = f(x,y) + g(x,y)H\left(\int_x^\infty \int_y^\infty h(s,t)W(u(s,t))dsdt\right),$$

then equation (2.22) can be restated as

$$u(x,y) \leq m(x,y) + \int_x^\infty a(s,y)u(s,y)ds. \tag{2.25}$$

Since $m(x,y)$ is positive, nondecreasing, we observe from (2.25) that

$$\frac{u(x,y)}{m(x,y)} \leq 1 + \int_x^\infty a(s,y)\frac{u(s,y)}{m(s,y)}ds. \tag{2.26}$$

Define

$$R(x,y)=1+\int_x^\infty a(s,y)\frac{u(s,y)}{m(s,y)}ds, \quad R(\infty,y)=1. \tag{2.27}$$

From (2.27) we have

$$\frac{\partial}{\partial x}R(x,y)=-a(x,y)\frac{u(x,y)}{m(x,y)}. \tag{2.28}$$

By using $\frac{u(x,y)}{m(x,y)}\le R(x,y)$ in (2.28), we obtain

$$\frac{\partial}{\partial x}R(x,y)\le a(x,y)R(x,y)$$

$$\text{i.e.,}\quad (\frac{\partial}{\partial x}R(x,y))/R(x,y)\le a(x,y).$$

By keeping y fixed in above inequality, setting $x=s$ and then integrating with respect to s from x to ∞, we have

$$R(x,y)\le \exp(\int_x^\infty a(s,y)ds)=\phi(x,y).$$

Thus (2.26) can be written as

$$u(x,y)\le \phi(x,y)[f(x,y)+g(x,y)H(\int_0^x\int_y^\infty h(s,t)W(u(s,t))dsdt)].$$

Again by following the same argument as in Theorem 2.2, we obtain our desired bound in (2.23).

Remark. There is no difficulty to obtain bounds for the integral inequalities of the form

$$\begin{aligned} u(x,y) &\le f(x,y)+\int_x^\infty a(s,y)G(u(s,y))ds \\ &\quad +g(x,y)H(\int_x^\infty\int_y^\infty h(s,t)W(u(s,t))dsdt) \end{aligned} \tag{2.29}$$

and

$$\begin{aligned} u(x,y) &\le f(x,y)+\int_x^\infty a(s,y)G(u(s,y))ds \\ &\quad +g(x,y)H(\int_0^x\int_y^\infty h(s,t)W(u(s,t))dsdt). \end{aligned} \tag{2.30}$$

Under some suitable conditions on the functions involved in these inequalities.

3. Application

In this section we shall give an application of Theorem 2.3 to obtain the bound on the solution of a nonlinear integral equation of the form

$$\begin{aligned} u(x,y) &= f(x,y) + \int_x^\infty K(x,y,s,u(s,y))ds \\ &\quad + F(x,y,\int_0^x \int_y^\infty \phi(s,y,s,t,u(s,t))dsdt) \end{aligned} \tag{3.1}$$

where all the functions involved in (3.1) are real-valued and defined on the respective domains of their definitions and such that

$$|K(x,y,s,u(x,y))| \leq a(s,y)|u(s,y)|, \tag{3.2}$$

$$|\phi(x,y,s,t,u)| \leq h(s,t)W(|u|), \tag{3.3}$$

$$|F(s,y,u)| \leq g(x,y)H(|u|), \tag{3.4}$$

where $h(x,y), g(x,y), a(x,y)$ are as in the assumptions of Theorem 2.3. Using (3.2), (3.3) and (3.4) in (3.1) we can easily have

$$\begin{aligned} |u(x,y)| &\leq |f(x,y)| + \int_x^\infty a(s,y)|u(s,y)|ds \\ &\quad + g(x,y)H(\int_0^x \int_y^\infty h(s,t)W(|u(s,t)|)dsdt). \end{aligned}$$

Applying Theorem 2.3, we obtain

$$\begin{aligned} |u(x,y)| &\leq \phi(x,y)[f(x,y) + g(x,y)H(G^{-1}(G\int_x^\infty \int_y^\infty h(s,t) \\ &\quad W(\phi(s,t)f(s,t))dsdt) \\ &\quad + \int_0^x \int_y^\infty h(s,t)W(\phi(s,t)f(s,t))dsdt))], \end{aligned} \tag{3.5}$$

where $\phi(x,y), G$ and G^{-1} are as defined in Theorem 2.3. Thus the right hand side of (3.5) gives an upper bound on the solution $u(x,y)$ of (3.1).

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