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# Minimal Complete Class of Generator Designs of Group Divisible Treatment Designs for Comparing Treatments with a Control

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## ABSTRACT

Bechhofer and Tamhane(1981) proposed Balanced Treatment Incomplete Block (BTIB) desings for comparing $p$ test treatments with a control treatment in blocks of size $k$. Notz and Tamhane(1983) solved the problem about determination of the minimal complete class for $k = 3$. However there are a number of design parameters for which BTIB designs do not exist. We suggest a new class of designs called Group Divisible Treatment Desings(GDTD's) that is a larger class including BTIB designs as a subclass. In this paper we give the minimal complete classes of generator designs for GDTD's with $k = 2$, $p \geq 4$(except prime number) and $k = 3$, $p = 4(2)6$.

## 1. Introduction

We consider the problem of comparing simultaneously several treatments, called test treatments, with a special treatment called the control treatment in blocks of size $k$. We shall use $0, 1, \cdots, p$ to label the $p + 1$ treatments being studied with 0 denoting the control treatmnt and $1, 2, \cdots, p$ denoting the p test treatments.

Bechhofer and Tamhane(1981) proposed BTIB designs for this problem. Notz and Tamhane(1983) studied minimal complete classes in BTIB designs. However

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there are a number of designs that do not exist in the class of BTIB designs. We propose a new class of designs, called GDTD.

Our main objective in this paper is to provide a study of Minimal Complete Class of Generator Designs(MCCGD's) in a class of GDTD's. The developments in the paper rely on a paper by Kim,K.H(1990).

For given $(p, k)$ a GDTD for any $b$ blocks can be constructed from a class of elementary GDTD's called generator designs. For given $(p, k)$ there exists a finite number of generator designs (see Theorem 3.1) but this number can be very large. However, it turns out that only a small subclass of these generator designs is sufficient in that essentially all admissible designs can be built from this class. We refer to this class as the minimal complete class. Admissible designs are important because they can be candidates for an optimal design. We show the minimal complete nature of the classes of generator designs that we have constructed for $k = 2,\ p \geq 4$ (except prime number) and $k = 3,\ p = 4(2)6$. The significance of these results is that the optimal designs can be built out of the generator designs in the minimal complete class.

## 2. Group Divisible Treatment Designs

Let $y_{ijh}$ be the response obtained by applying treatment $i$ to the $h$th plot of $j$th block; then the usual additive linear model(no treatment x block interaction) is

$$y_{ijh} = \mu + \alpha_i + \beta_j + \epsilon_{ijh}$$

where $\mu$ denotes the general mean, $\alpha_i$ the effect due to the $i$th treatment, $\beta_j$ the effect due to the $j$th block and the $\epsilon_{ijh}'s$ are uncorrelated random errors with mean 0 and variance $\sigma^2$ $(0 \leq i \leq p,\ \ 1 \leq j \leq b,\ \ 1 \leq h \leq r_{ij})$. The quantity $r_{ij}$, possibly equal to zero, denotes the number of replications of the $i$th treatment in the $j$ $th$ block. We will use $b$ to denote the number of blocks, while $k$ will be used to denote the common size of each block. We consider only connected GDTD's for the contrasts $\alpha_0 - \alpha_i$ $(1 \leq i \leq p)$ are estimable. Let $\widehat{\alpha_0} - \widehat{\alpha_i}$ be the BLUE of $\alpha_0 - \alpha_i$ $(1 \leq i \leq p)$. Then we have the following definition.

**Definition 2.1.** For given $(p, k, b)$ a GDTD is a proper block design for which the $p$ test treatments can be partitioned into $m$ groups of size $n$ such that

(i) $\sum_{j=1}^{b} r_{0j} r_{ij}$,

(ii) $\sum_{j=1}^{b} r_{ij} r_{i'j}$, $i$ and $i'$ are in the same group, $i \neq i'$,

and

(iii) $\sum_{j=1}^{b} r_{ij} r_{i'j}$, $i$ and $i'$ are in different groups,

do not depend on $i$ and $i'$, $i, \ i' = 1, 2, \cdots, p$.

Let $\lambda_{ii'} = \sum_{j=1}^{b} r_{ij} r_{i'j}$ $(i \neq i', \ \ 0 \leq i, \ i' \leq p)$. The following theorem states the necessary and sufficient conditions that a design must satisfy in order to be a GDTD. The proof of this theorem is given in Kim,K.H.(1987).

**Theorem 2.1.** For given $(p, k, b)$ a design is GDTD iff
$\lambda_{01} = \lambda_{02} = \cdots = \lambda_{0p} = \lambda_0$ (say)
$\lambda_{ii'} = \lambda_1$ (say) ($i$ and $i'$ are in the same group, $i \neq i'$)
and
$\lambda_{ii'} = \lambda_2$ (say) ($i$ and $i'$ are in different groups ).

Furthermore,

$$Var(\widehat{\alpha_0} - \widehat{\alpha}_i) = \frac{1}{m\,n}\left[\frac{1}{e_1} + \frac{m(n-1)}{e_2} + \frac{(m-1)}{e_3}\right]\sigma^2$$
$$= c^2\sigma^2 \quad (say) \quad (1 \leq i \leq p),$$

$$corr(\widehat{\alpha_0} - \widehat{\alpha}_i, \ \widehat{\alpha_0} - \widehat{\alpha_{i'}}) = \left[\frac{1}{e_1} - \frac{m}{e_2} + \frac{m-1}{e_3}\right] \Big/ \left[\frac{1}{e_1} + \frac{m(n-1)}{e_2} + \frac{m-1}{e_3}\right]$$
$$= \rho_1 \quad (say) \quad (i \ and \ i' \ are \ in \ the \ same \ group, \ \ i \neq i')$$

and

$$corr(\widehat{\alpha_0} - \widehat{\alpha}_i, \widehat{\alpha_0} - \widehat{\alpha_{i'}}) = \left[\frac{1}{e_1} - \frac{1}{e_3}\right] \Big/ \left[\frac{1}{e_1} + \frac{m(n-1)}{e_2} + \frac{m-1}{e_3}\right]$$
$$= \rho_2 \quad (say) \quad (i \ and \ i' \ are \ in \ different \ groups)$$

where $e_1 = \lambda_0/k$, $e_2 = (\lambda_0 + n\lambda_1 + (m-1)n\lambda_2)/k$, $e_3 = (\lambda_0 + mn\lambda_2)/k$ and the parameters $c^2$, $\rho_1$ and $\rho_2$ depend on the design employed. Clearly, for a GDTD to be implementable we must have $\lambda_0 > 0$. Also note that if $\lambda_1 = \lambda_2$ then these designs are called BTIB designs.

## 3. Minimal Complete Classes of Generator Designs for GDTD

we introduce the definition of a generator design for GDTD

**Definition 3.1.** For given $(p,k)$ a generator design is a GDTD such that no proper subclass of its blocks forms a GDTD on the same group structure and none of its blocks contains only one of the $p+1$ treatments.

Generator designs are important for the construction of GDTD's, as any such design is either a generator design or a union of copies of generator designs. In fact, if for given $(p,k)$ there are $''n''$ generator designs $D_i$ $(1 \le i \le n)$ where $D_i$ has parameters $(b_i,\ \lambda_0^{(i)},\ \lambda_1^{(i)},\ \lambda_2^{(i)})$, $i = 1,2,\cdots,n$; then GDTD $D = \bigcup_{i=1}^{n} f_i D_i$, obtained by taking union of $f_i \ge 0$ replications of $D_i$ on the same group structure has parameters $(b,\ \lambda_0,\ \lambda_1,\ \lambda_2)$ given by,

$$b = \sum_{i=1}^{n} f_i b_i,\ \lambda_0 = \sum_{i=1}^{n} f_i \lambda_0^{(i)},\ \lambda_1 = \sum_{i=1}^{n} f_i \lambda_1^{(i)},\ \lambda_2 = \sum_{i=1}^{n} f_i \lambda_2^{(i)}.$$

For $p \ge 4$ (except prime number) and $k = 2$ there are exactly five relevant generator designs for GDTD's (see Theorem 5.1 of the present paper). But it is not clear whether for any $(p,k)$ there are only finitely many generator designs. This question is answered in the proof of Theorem 3.1. This theorem is proven by the following two lemmas which are self-evident. First we give the notation used in the lemmas: For given $(p,k)$ let $S$ be the set of all distinct blocks which can be used in a GDTD and let us label these blocks in same mannar $1,2,\cdots,s$; $S$ consists of all samples of size $k$ with replacement from integers $0,1,\cdots,p$ except those $p+1$ samples of the type $(i,i,\cdots i)$ for $0 \le i \le p$. Next index the pairs $(0,1)$, $(0,2)$, $\cdots$, $(0,p)$, $(1,2)$, $(1,3)$, $\cdots$, $(p-1,p)$ by $1,2,\cdots$, $t = p(p+1)/2$. We relabel the pairs $(i,i')$ $(0 \le i \le p)$ in order of the pairs $(i,i')$ ($i$ and $i'$ are in the same group, $i \ne i'$) and the pairs $(i,i')$ ($i$ and $i'$ are in different groups). The numbers of these are $p(n-1)/2$ and $p(p-n)/2$ respectively. Let $\underline{Q}= \{Q_{gh}\}$ be a t x s matrix where $Q_{gh} = r_{ih}r_{i'h}$ $(o \leq i \leq i' \leq p)$, $g$ is the index of the pair $(i, i')$ $(1 \leq g \leq t)$, and $h$ is the index of the block $(1 \leq h \leq s)$. Then any design can be represented by a $S-vector$ $\underline{w}= (w_1, w_2, \cdots, w_s)$ where $w_h \geq 0$ is the frequency of the $h$th block in the design $(1 \leq h \leq s)$ and $b = \sum_{h=1}^{s} w_h$.

**Lemma 3.1.** A design with a frequency $\underline{w}$ is GDTD iff

$$\underline{w}Q' = \underline{\lambda}' = (\underbrace{\lambda_0, \cdots, \lambda_0}_{p}, \underbrace{\lambda_1, \cdots, \lambda_1}_{\frac{p(n-1)}{2}}, \underbrace{\lambda_2, \cdots, \lambda_2}_{\frac{p(p-n)}{2}}) \quad (3.1)$$

for some integers $\lambda_0 \geq 0$, $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ but not all zero.

**Lemma 3.2.** If $\underline{w}^{(1)}$ and $\underline{w}^{(2)}$ are two GDTS's with $w_h^{(2)} \geq w_h^{(1)}$ for $h = 1, 2, \cdots, s$ (with a strict inequality for at least some $h$ denoted by $\underline{w}^{(2)} > \underline{w}^{(1)}$) on the same group structure then $\underline{w}=\underline{w}^{(2)}$ - $\underline{w}^{(1)}$ is also a GDTD.

**Theorem 3.1.** For given $(p, k)$ there exist only finitely many generator designs.

**Proof.** Suppose that the theorem is not true. Then there exists an infinite sequence of generator designs $\underline{w}^{(1)}$, $\underline{w}^{(2)}$,$\cdots$. Choose a subsequence { $\underline{w}^{(i_j)}$ } from this sequence with the property $\underline{w}^{(i_{j+1})} > \underline{w}^{(i_j)}$; such a subsequence can always be chosen. Then from Lemma 3.2 it follows that $\underline{w}^{(i_{j+1})}$-$\underline{w}^{(i_j)}$ is a GDTD design. Therefore, $\underline{w}^{(i_{j+1})}$ is not a generator design. Thus we have reached a contradiction which proves the theorem.

The representation (3.1) can be used to construct GDTD's in particular generator designs for at least small values of $p$ and $k$. Many of the generator designs given in the present paper were constructed by using (3.1): the rest were constructed by using the methods of Kim.K.H(1990). To employ (3.1) it is first necessary to know the feasible values for the triple $(\lambda_0, \lambda_1, \lambda_2)$; these feasible values are obtained from Lemmas 4.2 and 4.3. Next for given $(\lambda_0, \lambda_1, \lambda_2)$ a lower bound on $b$ is obtained from Lemma 4.1.

Now we define the concepts of inadmissible and admissible designs. These concepts are motivated by the problem of joint confidence interval estimation of the $\alpha_0 - \alpha_i$ by Kim.K.H(1990).

**Definition 3.2.** Suppose that for given $(p,k)$ we have two GDTD's $D_1$ and $D_2$ with parameters $(b_1, \lambda_0^{(1)}, \lambda_1^{(1)}, \lambda_2^{(1)})$ and $(b_2, \lambda_0^{(2)}, \lambda_1^{(2)}, \lambda_2^{(2)})$. $D_2$ is inadmissible with respect to (wrt) $D_1$ iff $b_1 \leq b_2$, $c_1^2 \leq c_2^2$, $\rho_1^{(1)} \geq \rho_1^{(2)}$ and $\rho_2^{(1)} \geq \rho_2^{(2)}$ with at least one inequality strict. If a design is not inadmissible then it is said to be admissible. If $b_1 = b_2$, $c_1^2 = c_2^2$, $\rho_1^{(1)} = \rho_1^{(2)}$ and $\rho_2^{(1)} = \rho_2^{(2)}$ (or equivalenty $b_1 = b_2$, $\lambda_0^{(1)} = \lambda_0^{(2)}$, $\lambda_1^{(1)} = \lambda_1^{(2)}$ and $\lambda_2^{(1)} = \lambda_2^{(2)}$) then $D_1$ and $D_2$ are equivalent. For given $(p,k)$ the condidates for an "optimal" design will be all admissible designs that can be constructed from a given class of generator designs. Now we give the definition of the minimal complete class of generator designs for GDTD's.

**Definition 3.3.** For given $(p,k)$ the smallest class of generator designs $\{D_i(1 \leq i \leq n)\}$ from which all admissible designs for that $(p,k)$ (except possibly any equivalent ones) can be constructed is called the minimal complete class of generator designs $(MCCGD's)$ for GDTD's.

To obtain the minimal complete class from given $(p,k)$ we proceed in two steps. In the first step we delete any equivalent generator designs. Futhermore, if the union of two or more generator designs yields an equivalent generator design, then we delete the latter design. In the second step we delete the so-called strongly(S-) inadmissible generator designs from the class of nonequivalent generator designs obtained in the first step. The concept of S-inadmissibility is defined as follows.

**Definition 3.4.** If for given $(p,k)$ we have two GDTD's $D_1$ and $D_2$ (not necessarily generator designs), we say that $D_2$ is S-inadmissible wrt $D_1$ if $D_2$ is inadmissible wrt $D_1$, and if for any arbitraryGDTD $D_3$ we have that $D_2 \cup D_3$ is inadmissible wrt $D_1 \cup D_3$. An easily verifiable sufficient condition for S-inadmissibility of $D_2$ wrt $D_1$ is that

$$b_1 \leq b_2, \quad \lambda_0^{(1)} = \lambda_0^{(2)}, \quad \lambda_1^{(1)} \geq \lambda_1^{(2)}, \quad \lambda_2^{(1)} \geq \lambda_2^{(2)} \tag{3.2}$$

with at least one inequality being strict. We use a special case of (3.2) namely

$$b_1 \leq b_2, \quad \lambda_0^{(1)} = \lambda_0^{(2)}, \quad \lambda_1^{(1)} = \lambda_1^{(2)}, \quad \lambda_2^{(1)} = \lambda_2^{(2)} \tag{3.3}$$

repeately in the sequel to decide whether a given design $D_2$ is S-inadmissible or equivalent wrt another design $D_1$.

## 4. Theoretical Results

We give the relations between the parameters of a GDTD.

**Lemma 4.1.** For given $(p,k)$ a GDTD D with parameters $(b,\lambda_0,\lambda_1,\lambda_2)$ satisfies the following ineqalities on b:

$$\frac{pe_4}{k} \le b \le \frac{pe_4}{2} \tag{4.1}$$
$$where \quad e_4 = (2\lambda_0 + (n-1)\lambda_1 + n(m-1)\lambda_2)/k - 1 \quad and \quad p = mn.$$

Furthermore, the lower inequality is an equality iff the design is binary (i.e. $r_{ij} = 0$ or 1, $0 \le i \le p$; $1 \le j \le b$).

**Proof.** Let $r_i$ denote the number of replications on the $i$th treatment, $r_i = \sum_{j=1}^{b} r_{ij}$ $(0 \le i \le p)$. From (4.3a) and $C^* = [\{\lambda_0 + (n-1)\lambda_1 + n(m-1)\lambda_2\}I - \lambda_1 B_1 - \lambda_2 B_2]\ /\ k$ of Kim,K.H.(1987).

We have

$$kr_0 = p\lambda_0 + \sum_{j=1}^{b} r_{0j}^2 \tag{4.2}$$

and

$$kr_i = \lambda_0 + (n-1)\lambda_1 + n(m-1)\lambda_2 + \sum_{j=1}^{b} r_{ij}^2 \quad (1 \le i \le p) \tag{4.3}$$

Adding (4.2) and (4.3) we obtain

$$\sum_{i=0}^{p} r_i = kb = \left[2p\lambda_0 + (n-1)p\lambda_1 + n(m-1)p\lambda_2 + \sum_{i=0}^{p}\sum_{j=1}^{b} r_{ij}^2\right] \Big/ k \tag{4.4}$$

subject to the restriction that the $r_{ij}$ are nonnegative intergers satisfying $\sum_{i=0}^{p} r_{ij} = k$ for $1 \le j \le b$, it can be easily verified that $\sum_{i=0}^{p}\sum_{j=1}^{b} r_{ij}^2$ is minimized when each

$r_{ij} = 0$ or 1 and it is maximized when for each $j$ there is a pair of treatments $i_1, i_2$ $(i \neq i_2,\ 0 \leq i_1,\ i_2 \leq p)$ such that $r_{i_1 j} = k-1$, $r_{i_2 j} = 1$ and $r_{ij} = 0$ for $i \neq i_1, i_2$ $(0 \leq i \leq p,\ \ 1 \leq j \leq b)$.

Furthermore, the minimum value of $\sum\sum r_{ij}^2$ is $kb$ which when substituted in (4.4) yields the lower bound on $b$ in (4.1). The maximum value of $\sum\sum r_{ij}^2$ is $b(k^2 - 2k + 2)$ which when substituted in (4.4) yields the upper bound on $b$ in (4.1).

If the design is binary in terms of test-treatments then the test-treatments are replicated equally in the design with a common replicate size, $r = \lambda_0 + (n-1)\lambda_1 + n(m-1)\lambda_2/k - 1$.

**Lemma 4.2.** When $k$ is odd, the quantities $p\lambda_0$ and $\lambda_0 + (n-1)\lambda_1 + n(m-1)\lambda_2$ must be even.

**Proof.** Let $b_{il}$ denote the number of blocks in which the $i$th treatment is replicated $l$ times $(0 \leq l \leq k-1,\ \ 0 \leq i \leq p)$.

Note that

$$r_i = \sum_{l=1}^{k-1} l b_{il}, \qquad \sum_{j=1}^{b} r_{ij}^2 = \sum_{l=1}^{k-1} l^2 b_{il}. \tag{4.5}$$

Substituting (4.5) in (4.2) and (4.3) we get

$$p\lambda_0 = \sum_{l=1}^{k-1} l(k-l) b_{0l} \tag{4.6}$$

and

$$\lambda_0 + (n-1)\lambda_1 + n(m-1)\lambda_2 = \sum_{l=1}^{k-1} l(k-l) b_{il} \quad (1 \leq i \leq p). \tag{4.7}$$

By noting that when $k$ is odd, the coefficients $l(k-l)$ are even for $1 \leq l \leq k-1$ the lemma follows.

**Lemma 4.3.** For any GDTD for $p \geq 4$ (except prime number), $k = 3$ we have

$$(n-1)\lambda_1 + n(m-1)\lambda_2 \ \geq\ \lambda_0 \tag{4.8}$$

if $\lambda_1 > 0$, $\lambda_2 > 0$ and if the design does not contain the generator design

$$\left\{ \begin{matrix} 0 & 0 & \ldots & 0 \\ 1 & 2 & \ldots & p \\ 1 & 2 & \ldots & p \end{matrix} \right\}.$$

**Proof.** The lemma follows trivially for $\lambda_0 = 0$. Thus assume that $\lambda_0 > 0$. we may change any blocks of the type $\begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix}$ for $i \geq 1$ to $\begin{pmatrix} 0 \\ i \\ i \end{pmatrix}$ without affecting $\lambda_0$ or $\lambda_1$ or $\lambda_2$. We may assume that for some $i \geq 1$ there are no blocks of the type $\begin{pmatrix} 0 \\ i \\ i \end{pmatrix}$ because otherwise the GDTD would contain the generator design $\left\{\begin{matrix} 0 & 0 & \cdots & 0 \\ 1 & 2 & \cdots & p \\ 1 & 2 & \cdots & p \end{matrix}\right\}$. For that particular $i$ we can write

$$
\begin{aligned}
&(n-1)\lambda_1 + n(m-1)\lambda_2 - \lambda_0 \\
&= \underbrace{\sum_{\substack{i'=1 \\ i' \neq i}}^{p} \sum_{j=1}^{b} r_{ij} r_{i'j}}_{(i \text{ and } i' \text{ are in the same group})} + \underbrace{\sum_{i'=1}^{p} \sum_{j=1}^{b} r_{ij} r_{i'j} - \sum_{j=1}^{b} r_{ij} r_{0j}}_{(i \text{ and } i' \text{ are in differrent groups})} \qquad (4.9) \\
&= \sum_{j=1}^{b} r_{ij} \Bigg( \underbrace{\sum_{\substack{i'=1 \\ i' \neq i}}^{p} r_{i'j}}_{(i \text{ and } i' \text{ are in the same group})} + \underbrace{\sum_{i'=1}^{p} r_{i'j} - r_{0j}}_{(i \text{ and } i' \text{ are in the same groups})} \Bigg) \\
&\geq 0.
\end{aligned}
$$

The last step of (4.9) follows because the summand is negative iff $r_{ij} = 2$, $r_{0j} = 1$ and $r_{i'j} = 0$ for $i \neq i'$, a possibility that is ruled out.

## 5. *MCCGD's* for $k = 2$ and $p \geq 4$ (except prime number)

**Theorem 5.1.** For $k = 2$ and $p \geq 4$ the MCCGD's consists of the $D_1$, $D_2$, $D_3$, $D_4$ and $D_5$ which are given below.

$D_1 \equiv (b = p,\ r_0 = p,\ r = 1,\ \lambda_0 = 1,\ \lambda_1 = 0,\ \lambda_2 = 0)$.

| Block | 1 | 2 | ⋯ | ⋯ | p |
|---|---|---|---|---|---|
| Plots 1 | 0 | 0 | ⋯ | ⋯ | 0 |
| Plots 2 | 1 | 2 | ⋯ | ⋯ | p |

$D_2 \equiv (m=2,\ b=\frac{np}{2},\ r_0=0,\ r=n,\ \lambda_0=0,\ \lambda_1=0,\ \lambda_2=1).$

| Block | 1 | 2 ⋯ | | | ⋯ | ⋯ ⋯ | | $\cdots\frac{np}{2}$ |
|---|---|---|---|---|---|---|---|---|
| Plots 1 | 1 | 1 ⋯1 | 2 | 2 ⋯2 | | | n | n ⋯n |
| Plots 2 | $n+1$ | $n+2\cdots$p | $n+1$ | $n+2\cdots$p | ⋯ | ⋯ | $n+1$ | $n+2\cdots$p |

$D_3 \equiv (m=2,\ b=\frac{(n-1)p}{2}, r_0=0,\ r=n-1,\ \lambda_0=0,\ \lambda_1=1,\ \lambda_2=0).$

| Block | 1⋯ | ⋯ ⋯⋯ | | ⋯ | ⋯ | ⋯⋯ $\frac{(n-1)p}{2}$ |
|---|---|---|---|---|---|---|
| Plots 1 | 1⋯1 | 2⋯2 | $n-1$ | $n+1\cdots n+1$ | $n+2\cdots n+2$ | $p$ − 1 |
| Plots 2 | 2⋯n | 3⋯n⋯⋯ | n | $n+2\cdots$ p | $n+3\cdots$ p | ⋯⋯p |

$D_4 \equiv (m=3,\ b=np,\ r_0=0,\ r=2n,\ \lambda_0=0,\ \lambda_1=0,\ \lambda_2=1).$

| Block | 1 | | ⋯ | | | | | | ⋯ | ⋯ ⋯ | | | ⋯ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Plots 1 | 1 | 1 | ⋯ | 1 | 2 | 2 | ⋯ | 2 | | | $n+1$ | $n+1$ | ⋯ | $n+$ |
| Plots 1 | $n+1$ | $n+2$ | ⋯ | p | $n+1$ | $n+2$ | ⋯ | p | ⋯ | ⋯ | $2n+1$ | $2n+2$ | ⋯ | $p$ |

| | | ⋯ | | ⋯ ⋯ | | | | ⋯ np |
|---|---|---|---|---|---|---|---|---|
| $n+2$ | $n+2$ | ⋯ | $n+2$ | | $2n$ | $2n$ | ⋯ | $2n$ |
| $2n+1$ | $2n+2$ | ⋯ | p | ⋯ ⋯ | $2n+1$ | $2n+2$ | ⋯ | p |

$D_5 \equiv (m=3,\ b=\frac{(n-1)p}{2},\ r_0=0,\ r=n-1,\ \lambda_0=0,\ \lambda_1=1,\ \lambda_2=0).$

| Block | 1⋯ | ⋯ ⋯⋯ | | ⋯ | ⋯ | ⋯⋯ | ⋯ | ⋯⋯ $\frac{(n-1)p}{2}$ |
|---|---|---|---|---|---|---|---|---|
| Plots 1 | 1⋯1 | 2⋯2 | $n-1$ | $n+1\cdots n+1$ | $n+2\cdots n+2$ | | $2n+1\cdots 2n+1$ | $p$ − |
| Plots 2 | 2⋯n | 3⋯n⋯⋯ | n | $n+2\cdots$ $2n$ | $n+3\cdots$ $2n$ | ⋯⋯ | $2n+2\cdots$ $p$ | ⋯⋯p |

**Proof.** For each $p$ and $m=2$, the above mentioned designs $D_1$, $D_2$ and $D_3$ (in case $m=3$, $D_1$, $D_4$ and $D_5$) are completely binary and are the only possible generator designs and since they are distinct and neither of them is S-inadmissible wrt the other one, they construct the MCCGD'S for $k = 2$ and that particular $p$.

## 6. *MCCGD's* for $k = 3$ and $p = 4(2)6$

In this section we give MCCGD'S for$k = 3$, $p = 4(2)6$ and prove the minimal complete nature of the class in each case. The method of proof in each case is the same and we outline the general method here.

To show that for given $(p, k)$, a class of generator designs $\{D_1, D_2, \cdots, D_n\}$ is minimal complete, we consider an arbitrary GDTD $D$ for that $(p, k)$ having parameters $(b, \lambda_0, \lambda_1, \lambda_2)$. Then for that $D$ we show that there exists a GDTD $D^* = \bigcup_{i=1}^{n} F_i D_i$ such that $\lambda_0^* = \lambda_0$, $\lambda_1^* = \lambda_1$, $\lambda_2^* = \lambda_2$ and $b^* \leq b$. Thus $D$ is either equivalent to or S-inadmissible wrt $D^*$ ((3.3) of section 3). The proof is completed by finally noting that the class $\{D_1, D_2, \cdots, D_n\}$ consists of nonequivalent generator designs, none of which is S-inadmissible wrt to any other ones or unions of any other ones and therefore that class is minimal complete.

In the Proofs below for given $(p, k)$ we must consider several cases depending on the values of $(\lambda_0, \lambda_1, \lambda_2)$; in each case we construct the desired $D^*$ such that $\lambda_0^* = \lambda_0$, $\lambda_1^* = \lambda_1$, $\lambda_2^* = \lambda_2$ and explain why $D^*$ requires the smallest possible number of blocks(by using Lemma 4.1) which implies that $b^* \leq b$.

**Theorem 6.1.** For $p = 4$, $k = 3$, $m = 2$ and $n = 2$ the MCCGD's is as given in Table 1.

**Proof.** Consider an arbitrary GDTD D with parameter $(b, \lambda_0, \lambda_1 \lambda_2)$ for $p = 4$, $k = 3$, $m = 2$ and $n = 2$.

**Case 1.** $\lambda_1 = 0$, $\lambda_2 = 0$: It is easy to verify that $D$ is equivalent to $D^* = f_1 D_1$ where $f_1 \geq 1$. Thus, henceforth assume that $\lambda_1$ or $\lambda_2$ must be greater than or equal to 1.

**Case 2.** $\lambda_0 \equiv 0 (\text{mod } 2)$, $\lambda_1 \equiv 0 (\text{mod } 2)$: By Lemma 4.3, $\lambda_1/2 + \lambda_2 - \lambda_0/2 \geq 0$ and by Lemma 4.2, $\lambda_0 + \lambda_1 + 2\lambda_2$ is even and hence when $\lambda_0 \equiv 0 (\text{mod } 2)$ and $\lambda_1 \equiv 0 (\text{mod } 2)$, $\lambda_1/2 + \lambda_2 - \lambda_0/2$ must be odd or even. If $\lambda_1/2 + \lambda_2 - \lambda_0/2 \equiv 0(\text{mod}$

**Table 1. MCCGD's for $p=4$, $k=3$ (m=2, n=2)**

| $D_i$ | Design | b | $\lambda_0^{(i)}$ | $\lambda_1^{(i)}$ | $\lambda_2^{(i)}$ |
|---|---|---|---|---|---|
| $D_1$ | 0 0 0 0<br>0 0 0 0<br>1 2 3 4 | 4 | 2 | 0 | 0 |
| $D_2$ | 0 0<br>1 2<br>3 4 | 2 | 1 | 1 | 0 |
| $D_3$ | 0 0 0 0<br>1 1 2 3<br>2 4 3 4 | 4 | 2 | 0 | 1 |
| $D_4$ | 0 0 1 1 2<br>1 3 2 3 2<br>2 4 4 4 3 | 5 | 1 | 1 | 2 |
| $D_5$ | 0 0 0 0 1 2<br>1 1 3 3 1 2<br>2 2 4 4 4 3 | 6 | 2 | 0 | 2 |
| $D_6$ | 1 2<br>1 2<br>3 4 | 2 | 0 | 2 | 0 |
| $D_7$ | 1 1 2<br>2 3 2<br>3 4 4 | 3 | 0 | 2 | 1 |
| $D_8$ | 1 1 1 2<br>2 2 3 3<br>3 4 4 4 | 4 | 0 | 2 | 2 |
| $D_9$ | 1 1 1 2 1 3<br>2 2 2 3 1 3<br>3 3 4 4 4 4 | 6 | 0 | 2 | 3 |

2) then let $f_3 = \lambda_0/2$, $f_9 = \frac{1}{4}\{\lambda_1/2 + \lambda_2 - \lambda_0/2\}$ and $D^* = f_3 D_3 \cup f_9 D_9$. $D^*$ requires the smallest possible number of blocks namely $4\lambda_0/3 + 2\lambda_1/3 + 4\lambda_2/3 + 2/3$ which is the next higher integer to the lower bound of $4\lambda_0/3 + 2\lambda_1/3 + 4\lambda_2/3$ on

$b$ given by Lemma 4.1. If $\lambda_1/2 + \lambda_2 - \lambda_0/2 \equiv 1$ $(mod\ 2)$ then let $f_3 = \lambda_0/2$, $f_8 = \frac{1}{2}\{\lambda_1/2 + \lambda_2 - \lambda_0/2 - 1\}$ and $D^* = f_3D_3 \cup f_8D_8$. $D^*$ requires the smallest possible numbers of bolcks since $D_3$ and $D_8$ are binary.

**Case 3.** $\lambda_0 \equiv 0(\text{mod } 2)$, $\lambda_0 \geq 4$ and $\lambda_1 \equiv 0(\text{mod } 2)$: By Lemma 4.3, $\lambda_1/2+\lambda_2-2-(\lambda_0-4)/2 \geq 0$ and by Lemma 4.2, $\lambda_1/2+\lambda_2-2-(\lambda_0-4)/2$ must be odd or even. If $\lambda_1/2+\lambda_2-2-(\lambda_0-4)/2 \equiv 1(\text{mod } 2)$ then let $f_2 = (\lambda_0-4)/2$, $f_3 = 1$, $f_4 = 1$, $f_5 = \frac{1}{2}\{\lambda_1/2+\lambda_2-2-(\lambda_0-4)/2-1\}$ and $D^* = f_2D_2 \cup f_3D_3 \cup f_4D_4 \cup f_5D_5$. $D^*$ requires the smallest possible number of blocks namely $4\lambda_0/3+2\lambda_1/3+4\lambda_2/3+1$ which is the next higher integer to the lower bound $4\lambda_0/3 + 2\lambda_1/3 + 4\lambda_2/3$ on $b$ given by Lemma 4.1.

**Case 4.** $\lambda_0 = 1$: When $\lambda_0 = 1$, $\lambda_1$ must be odd by Lemma 4.2 and hence $\lambda_1 \geq 1$. Therefore $\lambda_1 - 1$ must be even. let $f_2 = 1$, $f_6 = (\lambda_1-1)/2$ and $D^* = f_2D_2 \cup f_6D_6$. $D^*$ requires the smallest possible number of blocks namely $4\lambda_0/3+2\lambda_1/3+4\lambda_2/3+2/3$ which is the next higher integer to the lower bound of $4\lambda_0/3 + 2\lambda_1/3 + 4\lambda_2/3$ on $b$ given by Lemma 4.1.

**Case 5.** $\lambda_0 \equiv 1(\text{mod } 2)$, $\lambda_1 \equiv 1(\text{mod } 2)$ : By Lemma 4.3, $(\lambda_1 - 1)/2 + \lambda_2 - (\lambda_0 - 1)/2 \geq 0$ and by Lemma 4.2, $(\lambda_1 - 1)/2 + \lambda_2 - (\lambda_0 - 1)/2$ is odd or even. If $(\lambda_1 - 1)/2 + \lambda_2 - (\lambda_0 - 1)/2 \equiv 0(\text{mod } 2)$ then let $f_2 = (\lambda_0 - 1)/2$, $f_3 = (\lambda_1 - 1)/2$, $f_7 = \frac{1}{2}\{(\lambda_1-1)/2+\lambda_2-(\lambda_0-1)/2\}$ and $D^* = f_2D_2 \cup f_3D_3 \cup f_7D_7$. $D^*$ requires the smallest opssible number of blocks namely $4\lambda_0/3+2\lambda_1/3+4\lambda_2/3+1/3$ which is the next higher integer to the lower bound of $4\lambda_0/3+2\lambda_1/3+4\lambda_2/3$ on $b$ given by Lemma 4.1. If $(\lambda_1 - 1)/2 + \lambda_2 - (\lambda_0 - 1)/2 \equiv 1(\text{mod } 2)$ then let $f_2 = 1$, $f_3 = (\lambda_0 - 1)/2$, $f_8 = \frac{1}{2}\{(\lambda_1-1)/2+\lambda_2-(\lambda_0-1)/2-1\}$ and $D^* = f_2D_2 \cup f_3D_3 \cup f_8D_8$. $D^*$ requires the smallest possible number of blocks since $D_2$, $D_3$ and $D_8$ are binary.

**Theorem 6.2.** For $p = 6$, $k = 3$, $m = 2$ and $n = 3$ the MCCGD's is as given in Table 2.

**Proof.** Consider an arbitrary GDTD D with parameters $(b, \lambda_0, \lambda_1, \lambda_2)$ for $p = 6$, $k = 3$, $m = 2$ and $n = 3$.

**Case 1.** $\lambda_1 = 0$, $\lambda_2 = 0$: It is easy to verify that D is equivalent to $f_1D_1$ for some $f_1 \geq 1$. Thus, henceforth assume that $\lambda_1$ or $\lambda_2$ must be $\geq 1$.

**Case 2.** $\lambda_0 \equiv 0(\text{mod } 3)$, $\lambda_1 \equiv 0(\text{mod } 3)$: By Lemma 4.3, $2\lambda_1/3+\lambda_2-\lambda_0/3 \geq 0$ and by Lemma 4.2, $\lambda_0 + 2\lambda_1 + 3\lambda_2$ is even and hence when $\lambda_0 \equiv 0(\text{mod } 3)$ and $\lambda_1 \equiv 0(\text{mod } 3)$, $2\lambda_1/3+\lambda_2-\lambda_0/3$ is even. Let $f_2 = \lambda_0/3$, $f_6 = \frac{1}{4}\{2\lambda_1/3+\lambda_2-\lambda_0/3\}$

**Table 2. MCCGD's for $p = 6$, $k = 3$, $m = 2$ and $n = 3$**

| Label | | | | | | | | | | | | | | | | | | | $b_i$ | $\lambda_0^{(i)}$ | $\lambda_1^{(i)}$ | $\lambda_2^{(i)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $D_1$ | 0 | 0 | 0 | 0 | 0 | 0 | | | | | | | | | | | | | 6 | 2 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 0 | | | | | | | | | | | | | | | | |
| | 1 | 2 | 3 | 4 | 5 | 6 | | | | | | | | | | | | | | | | |
| $D_2$ | 0 | 0 | 0 | 0 | 0 | 0 | | | | | | | | | | | | | 6 | 2 | 1 | 0 |
| | 1 | 1 | 3 | 2 | 2 | 4 | | | | | | | | | | | | | | | | |
| | 3 | 5 | 5 | 4 | 6 | 6 | | | | | | | | | | | | | | | | |
| $D_3$ | 0 | 0 | 0 | 1 | 2 | 2 | 1 | | | | | | | | | | | | 7 | 1 | 1 | 1 |
| | 1 | 3 | 5 | 4 | 3 | 4 | 3 | | | | | | | | | | | | | | | |
| | 2 | 4 | 6 | 6 | 6 | 5 | 5 | | | | | | | | | | | | | | | |
| $D_4$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | | | | | | | | | | 9 | 3 | 0 | 1 |
| | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | | | | | | | | | | | | | |
| | 2 | 4 | 6 | 3 | 5 | 4 | 6 | 5 | 6 | | | | | | | | | | | | | |
| $D_5$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | | | | | | | 12 | 2 | 1 | 2 |
| | 1 | 1 | 3 | 3 | 5 | 5 | 3 | 4 | 4 | 3 | 3 | 4 | | | | | | | | | | |
| | 2 | 2 | 4 | 4 | 6 | 6 | 6 | 5 | 6 | 5 | 6 | 5 | | | | | | | | | | |
| $D_6$ | 0 | 0 | 0 | 1 | 1 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 3 | 3 | | | | 15 | 1 | 2 | 3 |
| | 1 | 3 | 5 | 4 | 4 | 3 | 3 | 4 | 4 | 2 | 3 | 2 | 5 | 4 | 5 | | | | | | | |
| | 2 | 4 | 6 | 6 | 6 | 6 | 6 | 5 | 5 | 3 | 4 | 5 | 6 | 5 | 6 | | | | | | | |
| $D_7$ | 1 | 2 | | | | | | | | | | | | | | | | | 2 | 0 | 1 | 0 |
| | 3 | 4 | | | | | | | | | | | | | | | | | | | | |
| | 5 | 6 | | | | | | | | | | | | | | | | | | | | |
| $D_8$ | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | | | | | | | | | 10 | 0 | 2 | 2 |
| | 2 | 2 | 3 | 4 | 5 | 3 | 4 | 4 | 4 | 5 | | | | | | | | | | | | |
| | 3 | 5 | 4 | 6 | 6 | 6 | 5 | 6 | 5 | 6 | | | | | | | | | | | | |
| $D_9$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 18 | 0 | 3 | 4 |
| | 2 | 2 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 3 | 3 | 3 | 4 | 5 | 4 | 4 | 5 | 5 | | | | |
| | 3 | 4 | 5 | 6 | 4 | 6 | 5 | 6 | 6 | 4 | 5 | 6 | 5 | 6 | 5 | 6 | 6 | 6 | | | | |

and $D^* = f_2D_2 \cup f_6D_6$. $D^*$ requires the smallest possible number of blocks since $D_2$ and $D_6$ are binary.

**Case 3.** $\lambda_0 \equiv 0 (\text{mod } 3)$, $\lambda_1 \equiv 1 (\text{mod } 3)$: By Lemma 4.3, $2(\lambda_1 - 1)/3 + \lambda_2 + 2 - \lambda_0/3 \geq 0$ and be Lemma 4.2, $2(\lambda_1 - 1)/3 + \lambda_2 + 2 - \lambda_0/3$ is even. Let $f_4 = \lambda_0/3$,

$f_7 = 1$, $f_9 = \frac{1}{8}\{2(\lambda_1 - 1)/3 + \lambda_2 + 2 - \lambda_0/3\}$ and $D^* = f_4D_4 \cup f_7D_7 \cup f_9D_9$. $D^*$ requires the smallest possible number of blocks since $D_4$, $D_7$ and $D_9$ are binary.

**Case 4.** $\lambda_0 \equiv 0(\text{mod } 3)$, $\lambda_1 \equiv 2(\text{mod } 3)$: By Lemma 4.3, $2(\lambda_1 - 2)/3 + \lambda_2 + 2 - \lambda_0/3 \geq 0$ and by Lemma 4.2, $2(\lambda_1 - 2)/3 + \lambda_2 + 2 - \lambda_0/3$ is even. Let $f_2 = \lambda_0/3$, $f_3 = 1$, $f_9 = \frac{1}{8}\{2(\lambda_1 - 2)/3 + \lambda_2 + 2 - \lambda_0/3\}$ and $D^* = f_2D_2 \cup f_3D_3 \cup f_9D_9$. $D^*$ requires the smallest possible number of blocks since $D_2$, $D_3$ and $D_9$ are binary.

**Case 5.** $\lambda_0 \equiv 1(\text{mod } 3)$, $\lambda_1 \equiv 0(\text{mod } 3)$: By Lemma 4.3, $2\lambda_1/3 + \lambda_2 - 1 - (\lambda_0 - 1)/3 \geq 0$ and by Lemma 4.2, $2\lambda_1/3 + \lambda_2 - 1 - (\lambda_0 - 1)/3$ must be even. Let $f_2 = 1$, $f_5 = (\lambda_0 - 1)/3$, $f_7 = \frac{1}{2}\{2\lambda_1/3 + \lambda_2 - 1 - (\lambda_0 - 1)/3\}$ and $D^* = f_2D_2 \cup f_5D_5 \cup f_7D_7$. $D^*$ requires the smallest possible number of blocks since $D_2$, $D_5$ and $D_7$ are binary.

**Case 6.** $\lambda_0 \equiv 2(\text{mod } 3)$, $\lambda_1 \equiv 0(\text{mod } 3)$: By Lemma 4.3, $2\lambda_1/3 + \lambda_2 - 2 - (\lambda_0 - 2)/3 \geq 0$ and by Lemma 4.2, $2\lambda_1/3 + \lambda_2 - 2 - (\lambda_0 - 2)/3$ must be even. Let $f_4 = (\lambda_0 - 2)/3$, $f_5 = 1$, $f_8 = \frac{1}{4}\{2\lambda_1/3 + \lambda_2 - 2 - (\lambda_0 - 2)/3\}$ and $D^* = f_4D_4 \cup f_5D_5 \cup f_8D_8$. $D^*$ requires the smallest possible number of blocks since $D_4$, $D_5$ and $D_8$ are binary.

**Thorem 6.3** For $p = 6$, $k = 3$, $m = 3$ and $n = 2$ the MCCGD's is as given in Table 3.

**Proof.** Consider an arbitrary GDTD D with parameters $(b, \lambda_0, \lambda_1, \lambda_2)$ for $p = 6$, $k = 3$, $m = 3$ and $n = 2$.

**Case 1.** $\lambda_1 = 0$, $\lambda_2 = 0$: It is easy to verify that D is equivalent to $f_1D_1$ for some $f_1 \geq 1$. Thus, henceforth assume that $\lambda_1$ or $\lambda_2$ must be $\geq 1$.

**Case 2.** $\lambda_0 \equiv 0(\text{mod } 4)$, $\lambda_1 \equiv 0(\text{mod } 4)$: By Lemma 4.3, $\lambda_1/4 + \lambda_2 - \lambda_0/4 \geq 0$.

If $\lambda_1/4 + \lambda_2 - \lambda_0/4 \equiv 0(\text{mod } 2)$ then let $f_2 = 2$, $f_3 = \lambda_0/4$, $f_6 = \frac{1}{2}\{\lambda_1/4 + \lambda_2 - \lambda_0/4\}$ and $D^* = f_2D_2 \cup f_3D_3 \cup f_6D_6$. $D^*$ requires the smallest possible number of blocks since $D_2$, $D_3$ and $D_6$ are binary. If $\lambda_1/4 + \lambda_2 - \lambda_0/4 \equiv 1(\text{mod } 2)$ then let $f_2 = 2$, $f_3 = 1$, $f_5 = \lambda_0/4$, $f_6 = \frac{1}{2}\{\lambda_1/4 + \lambda_2 - \lambda_0/4 - 1\}$ and $D^* = f_2D_2 \cup f_3D_3 \cup f_5D_5 \cup f_6D_6$. $D^*$ requires the smallest possible number of blocks since $D_2$, $D_3$, $D_5$ and $D_6$ are binary.

**Case 3.** $\lambda_0 \equiv 1(\text{mod } 4)$, $\lambda_1 \equiv 1(\text{mod } 4)$: By Lemma 4.3, $(\lambda_1 - 1)/4 + \lambda_2 - (\lambda_0 - 1)/4 \geq 0$.

If $(\lambda_1 - 1)/4 + \lambda_2 - (\lambda_0 - 1)/4 \equiv 0(\text{mod } 3)$ then let $f_2 = (\lambda_0 - 1)/4$, $f_6 = 2$, $f_4 = \frac{1}{3}\{(\lambda_1 - 1)/4 + \lambda_2 - (\lambda_0 - 1)/4\}$ and $D^* = f_2D_2 \cup f_6D_6 \cup f_4D_4$. $D^*$

**Table 3. MCCGD's for $p=6$, $k=3$, $m=3$ and $n=2$**

| Label | | $b_i$ | $\lambda_0^{(i)}$ | $\lambda_1^{(i)}$ | $\lambda_2^{(i)}$ |
|---|---|---|---|---|---|
| $D_1$ | 0 0 0 0 0 0 | 6 | 2 | 0 | 0 |
| | 0 0 0 0 0 0 | | | | |
| | 1 2 3 4 5 6 | | | | |
| $D_2$ | 0 0 0 | 3 | 1 | 1 | 0 |
| | 1 2 3 | | | | |
| | 4 5 6 | | | | |
| $D_3$ | 0 0 0 0 0 0 1 4 | 8 | 2 | 0 | 1 |
| | 1 1 2 2 3 3 2 5 | | | | |
| | 5 6 4 6 4 5 3 6 | | | | |
| $D_4$ | 0 0 0 0 0 0 0 0 0 0 0 0 | 12 | 4 | 0 | 1 |
| | 1 1 1 1 2 2 2 3 3 4 4 5 | | | | |
| | 2 3 5 6 3 4 6 4 5 5 6 6 | | | | |
| $D_5$ | 1 1 2 3 | 4 | 0 | 0 | 1 |
| | 2 5 4 4 | | | | |
| | 3 6 6 5 6 6 | | | | |
| $D_6$ | 1 1 1 2 2 3 | 6 | 0 | 2 | 1 |
| | 2 4 3 3 5 4 | | | | |
| | 4 5 6 5 6 6 | | | | |

requires the smallest possible number of blocks since $D_2$, $D_6$ and $D_4$ are binary. If $(\lambda_1-1)/4+\lambda_2-(\lambda_0-1)/4\equiv 1(\text{mod } 3)$ then let $f_2=(\lambda_0-1)/4$, $f_4=1$, $f_6=2$, $f_5=\frac{1}{3}\{(\lambda_1-1)/4+\lambda_2-(\lambda_0-1)/4-1\}$ and $D^*=f_2D_2\cup f_4D_4\cup f_6D_6\cup f_5D_5$. $D^*$ requires the smallest possible number of blocks since $D_2$, $D_4$, $D_6$ and $D_5$ are binary. If $(\lambda_1-1)/4+\lambda_2-(\lambda_0-1)/4\equiv 2(\text{mod } 3)$ then let $f_2=(\lambda_0-1)/4$, $f_5=2$, $f_6=2$, $f_4=\frac{1}{3}\{(\lambda_1-1)/4+\lambda_2-(\lambda_0-1)/4-2\}$ and $D^*=f_2D_2\cup f_5D_5\cup f_6D_6\cup f_4D_4$. $D^*$ requires the smallest possible number of blocks since $D_2$, $D_5$, $D_6$ and $D_4$ are binary.

**Case 4.** $\lambda_0\equiv 2(\text{mod } 4)$, $\lambda_1\equiv 0(\text{mod } 4)$: By Lemma 4.3, $\lambda_1/4+\lambda_2-1-(\lambda_0-2)/4\geq 0$.

If $\lambda_1/4+\lambda_2-1-(\lambda_0-2)/4\equiv 0(\text{mod } 2)$ then let $f_2=2$, $f_4=1$, $f_5=(\lambda_0-2)/4$, $f_6=\frac{1}{2}\{\lambda_1/4+\lambda_2-1-(\lambda_0-2)/4\}$ and $D^*=f_2D_2\cup f_4D_4\cup f_5D_5\cup f_6D_6$. $D^*$

requires the smallest possible number of blocks since $D_2$, $D_4$, $D_5$ and $D_6$ are binary. If $\lambda_1/4+\lambda_2-1-(\lambda_0-2)/4 \equiv 1(\mathrm{mod}\ 2)$ then let $f_2=1$, $f_4=(\lambda_0-2)/4$, $f_5=2$, $f_6=\{\lambda_1/4+\lambda_2-2-(\lambda_0-2)/4\}$ and $D^*=f_2D_2\cup f_4D_4\cup f_5D_5\cup f_6D_6$. $D^*$ requires the smallest possible number of blocks since $D_2$, $D_4$, $D_5$ and $D_6$ are binary.

## 7. Discussion

The class of GDTD's contains the designs that do not exist in the class of BTIB designs. The class of these designs is larger than the class of BTIB designs and provides us with designs that improve on the optimal designs in thier class. But determination of the MCCGD's in GDTD's for given$(p,k,m,n)$ was an open problem. We solve the problem for $k=2,3$. The other values of $p$ and $k$ is of less practical interest. The optimal designs can be constructed by using the generator designs in the minimal complete class for that case.

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