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Numerical Quadrature Techniques for Inverse Fourier Transform in Two-Dimensional Resistivity Modeling

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ABSTRACT : This paper compares numerical quadrature techniques for computing an inverse Fourier transform integral in two-dimensional resistivity modeling. The quadrature techniques using exponential and cubic spline interpolations are examined for the case of a homogeneous earth model. In both methods the integral over the interval from 0 to λ_{min}, where λ_{min} is the minimum sampling spatial wavenumber, is calculated by approximating Fourier transformed potentials to a logarithmic function. This scheme greatly reduces the inverse Fourier transform error associated with the logarithmic discontinuity at $\lambda=0$. Numrical results show that, if the sampling intervals are adequate, the cubic spline interpolation method is more accurate than the exponential interpolation method.

INTRODUCTION

Numerical analysis using the finite difference (Dey and Morrison, 1979b) or finite element method (Pridmore et al., 1981) was performed to solve three-dimensional (3D) resistivity problems. However, these 3D models require extensive computing time and memory storages so that, in many cases, more practical two-dimensional (2D) models were applied using finite element (Coggon, 1971) and finite difference methods (Dey and Morrison, 1979a). These 2D numerical methods employ a Fourier cosine transform because a point source of current over 2D resistivity structure constitutes a 3D potential field. Therefore, an inverse Fourier transformation must be performed to obtain the potential field in cartesian space.

The inverse Fourier transform can be computed by numerical quadrature techniques using exponential (Dey and Morrison, 1979a) or cubic spline interpolation (Tripp et al., 1984). Since these numerical quadratures are simple and fast, they are preferable to implement 2D models. However, a detailed discussion for these techniques has not been given. This paper presents some error estimations of these methods for the case of a homogeneous earth model.

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DC RESISTIVITY RESPONSES

DC (direct current) resistivity response for 2D conductivity structure, $\sigma(x, z)$, is described by (Grant and West, 1965).

$$-\nabla\cdot[\sigma(x,z)\nabla\phi(x,y,z)]=i_s(x,y,z). \qquad (1)$$

where $\phi(x, y, z)$ is the electrical potential, and $i_s(x, y, z)$ is the source current distribution. Note that although the conductivity structure is 2D, the current source is 3D (point). By taking a Fourier transformation of equation (1) with respect to the y-coordinate, we obtain

$$-\nabla\cdot[\sigma(x,z)\nabla\Phi(x,\lambda,z)]+\lambda^2\sigma(x,z)\Phi(x,\lambda,z)=I_s(x,\lambda,z), \qquad (2)$$

where λ is the Fourier transform variable. Coggon (1971) solved equation (2) for $\Phi(x, \lambda, z)$ using the finite element method, and Dey and Morrison (1979 a) employed the finite difference approximation.

To calculate the inverse Fourier transform and obtain the electrical potential $\phi(x, y, z)$, we must solve equation (2) for various different values of λ. The inverse Fourier transformation of $\Phi(x, \lambda, z)$ is given by (Dey and Morrison, 1979a)

$$\phi(x,y,z)=\frac{2}{\pi}\int_0^\infty \Phi(x,\lambda,z)\cos(\lambda y)d\lambda. \qquad (3)$$

When both the source and the field points are located in a plane of $y=0$, the transformation is reduced to

$$\phi(x, 0, z) = \frac{2}{\pi}\int_0^\infty \Phi(x, \lambda, z)d\lambda. \quad (4)$$

Since the Fourier transformed potential $\Phi(x, \lambda, z)$ is, in general, a monotonically decreasing function (Pelton et al., 1978), the behavior of the function can be adequately predicted over its entire range using several values of λ.

INVERSE FOURIER TRANSFORM

Dey and Morrison (1979a) evaluated the integration term in equation (4) using a subsectional exponential interpolation of the integrand and its analytic solution for each subintegral. Fitting the envelope of $\Phi(\lambda)$ in each subsection $\lambda_1 \leq \lambda \leq \lambda_2$ by an exponential function b exp$(-a\lambda)$ yields

$$b\int_{\lambda_1}^{\lambda_2} e^{-a\lambda}d\lambda = \frac{\Phi_1 - \Phi_2}{a}, \quad (5)$$

where

$$a = \frac{\ln(\Phi_1/\Phi_2)}{\lambda_2 - \lambda_1}. \quad (6)$$

Dey and Morrison (1979a) performed the integration in equation (4) by obtaining the cumulative sum of these subsectional integrals up to a sufficiently large λ_{max}, where λ_{max} is the maximum sampling value of λ. Note that the integral over the interval $[0, \lambda_{min}]$ must be added to the cumulative sum, where λ_{min} is the minimum sampling value of λ.

The simplest method to obtain the integral over the interval $[0, \lambda_{min}]$ is the rectangle rule:

$$\int_0^{\lambda min} \Phi(\lambda)d\lambda \approx \lambda_{min}\Phi(\lambda_{min}). \quad (7)$$

However, this approximation is not accurate because $\Phi(\lambda)$ approaches infinity as λ approaches to zero. A logarithmic discontinuity at $\lambda=0$ causes significant error in the inverse Fourier transform (Zhao et al., 1986).

For small arguments of λ the Fourier transformed potential $\Phi(\lambda)$ can be approximated using a logarithmic function (Wait, 1987). Replacing $\Phi(\lambda)$ by $-b\ln(a\lambda)$ gives

$$-b\int_0^{\lambda min} \ln(a\lambda)d\lambda = \lambda_{min}[\Phi(\lambda_{min}) + b]. \quad (8)$$

The constant b may be obtained by fitting the envelope of $\Phi(\lambda)$ with the same logarithmic function for the subsection $\lambda_{min} \leq \lambda \leq \lambda_{min+1}$, i.e.,

$$b = \frac{\Phi(\lambda_{min}) - \Phi(\lambda_{min+1})}{\ln(\lambda_{min+1}/\lambda_{min})}. \quad (9)$$

Another numerical quadrature technique for evaluating the integral in equation (4) was given by Tripp et al. (1984). The technique employs an analytic integration of a cubic spline interpolant of sampled values of $\Phi(\lambda)$. However, a detailed explanation of their technique was not given.

If the envelope of $\Phi(\lambda)$ in the each subsection $\lambda_1 \leq \lambda \leq \lambda_2$ is fitted by a cubic polynomial

$$\Phi(\lambda) = \sum_{j=1}^{4} c_j(\lambda - \lambda_1)^{j-1}, \quad (10)$$

then

$$\int_{\lambda_1}^{\lambda_2} \Phi(\lambda)d\lambda = \sum_{j=1}^{4} \frac{c_j}{j}(\lambda_2 - \lambda_1)^j, \quad (11)$$

where c_j is the polynomial coefficients. If the boundary derivatives of $\Phi(\lambda)$ at λ_{min} and λ_{max} are given, all of the coefficients c_j can be determined (e.g., Conte and deBoor, 1980).

The two boundary slopes can be estimated by considering the asymptotic nature of $\Phi(\lambda)$. for small λ the Fourier transformed potential $\Phi(\lambda)$ is approximated by a logarithmic function:

$$\Phi(\lambda) \approx -b\ln(\lambda). \quad (12)$$

Hence,

$$\left.\frac{\partial\Phi(\lambda)}{\partial\lambda}\right|_{\lambda_{min}} \approx \frac{-b}{\lambda_{min}} \quad (13)$$

For large λ, on the other hand, the potential $\Phi(\lambda)$ can be approximated by an exponential function (Pelton et al., 1978):

$$\Phi(\lambda) \approx \exp(-c\lambda) \quad (14)$$

Then

$$\left.\frac{\partial\Phi(\lambda)}{\partial\lambda}\right|_{\lambda_{max}} \approx -c\Phi(\lambda_{max}), \quad (15)$$

where a constant c may be evaluated as

$$c = \frac{\ln[\Phi(\lambda_{max-1})/\Phi(\lambda_{max})]}{\lambda_{max} - \lambda_{max-1}}. \quad (16)$$

HOMOGENEOUS EARTH MODEL

In this section I investigate the accuracy of the numerical quadrature schemes described above for the case of a homogeneous earth model. In the homogeneous earth the Fourier transformed potential

$\Phi(\lambda)$ due to a point source of current located at the earth's surface is

$$\Phi(\lambda)=(I\rho/2\pi)K_o(\lambda r), \quad (17)$$

where

$$r=(x^2+z^2)^{1/2},$$

I is the strength of the induced current and ρ is the resistivity of the earth, and K_o is the modified Bessel function of order 0. A definite integral of the zero-order modified Bessel function is (Abramowitz and Stegun, 1970; 11. 1. 9)

$$\int_o^\lambda K_o(t)dt=-\left(\gamma+\ln\frac{\lambda}{2}\right)\lambda\sum_{k=0}^{\infty}\frac{(\lambda/2)^{2k}}{(k!)^2(2k+1)}$$
$$+\lambda\sum_{k=1}^{\infty}\frac{(\lambda/2)^{2k}}{(k!)^2(2k+1)^2}$$
$$+\lambda\sum_{k=1}^{\infty}\frac{(\lambda/2)^{2k}}{(k!)^2(2k+1)}\left(1+\frac{1}{2}+\cdots\frac{1}{k}\right), \quad (18)$$

where $\gamma=0.5772\cdots$ is the Euler's constant. The semi-infinite integral of the modified Bessel function can be also found in Abramowitz and Stegun (1970, 11. 4. 23):

$$\int_o^\infty K_o(t)dt=\frac{\pi}{2}. \quad (19)$$

Substituting equation (19) for equation (4) yields a well-known result for the uniform half-space as

$$\phi(r)=I\rho/2\pi r. \quad (20)$$

Figure 1 shows the Fourier transformed potentials due to a single current source calculated using equation (17) for $I=1$ A and $\rho=1\,\Omega\cdot$m. To perform the integration in equation (4), I sampled the Fourier transformed potentials for 10 values of λ_i:

$\lambda_i=0.005$, 0.01, 0.02, 0.04, 0.08, 0.16, 0.32, 0.64, 1.28 and 2.56.

With these sampled values $\Phi(\lambda_i)$, I computed the electrical potential $\phi(r)$ using the exponential and cubic spline interpolation methods. Table 1 lists the numerical results for the case of $r=4$ m. The analytic results in Table 1 are obtained using equation (18).

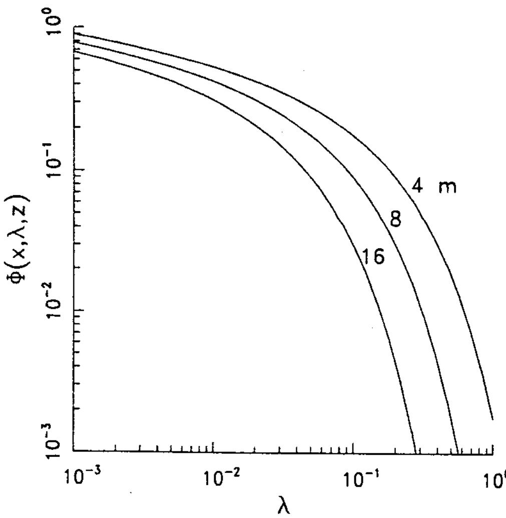

Fig. 1. Fourier transformed potentials for $r=4$, 8, 12, 16 and 20 m.

The exponential interpolation method causes overshoot, while the cubic spline method causes slight undershoot. Note that the numerical result for $\lambda_{max}=0.005$ is computed using equation (8). If equation (7) is employed, the result becomes 2.041 mV and this value is about 25% lower than the analytic solution of 2.547 mV.

Table 2 shows the relative errors of the numerical quadrature techniques. The relative error, RE (%), is calculated by

$$\text{RE }(\%)=(\phi_n-\phi_a)/\phi_a\times 100,$$

where ϕ_n and ϕ_a represent the potentials that are estimated numerically and analytically, respectively. From the results listed in Tables 1 and 2, I can conclude that the numerical quadrature technique using a cubic spline interpolation is more accurate than that using an exponential interpolation.

The sampling interval also affects the accuracy of the numerical quadrature, and the numerical error increases with a decrease in the number of λ. If the eight values of λ that are equally spaced logarithmically from 0.005 to 2.56 are used, for example, then the relative errors (RE) for $r=4$ m are 2.016% in the exponential interpolation and -1.193% in the cubic spline interpolation, respectively. It should be noted that the cubic spline interpolation

Table 1. Cumulative potentials (mV) for various λ_{max}.

λ_{max}	Potential (mV)		
	Exponential	Cubic spline	Analytic
0.005	2.5465	2.5465	2.5473
0.01	4.4066	4.3907	4.3927
0.02	7.4247	7.3810	7.3838
0.04	12.067	11.971	11.978
0.08	18.626	18.449	18.461
0.16	26.667	26.367	26.394
0.32	34.332	33.893	33.944
0.64	38.965	38.380	38.502
1.28	40.197	39.512	39.713
2.56	40.273	39.677	39.788

Table 2. Relative errors (%) of numerical quadrature techniques.

r (m)	Relative error (%)	
	Exponential	Cubic spline
4	1.218	−0.2800
8	1.171	−0.2760
12	1.113	−0.3514
16	1.038	−0.3445
20	0.947	−0.3944

method may fail to predict the accurate interpolant, particularly in the range of small $\Phi(\lambda)$, when the sampling interval becomes large.

CONCLUDING REMARKS

In this paper I compared the numerical quadrature techniques using exponential and cubic spline interpolations for computing the inverse Fourier transform in 2D resistivity modeling. If the sampling intervals are adequate, the cubic spline interpolation method is more accurate than the exponential interpolation method. In both methods the integral over the interval $[0, \lambda_{min}]$ must be evaluated according to equation (8) instead of equation (7). Equation (8) greatly reduces the inverse Fourier transform error associated with the logarithmic discontinuity at $\lambda=0$.

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2차원 전기비저항 모델링에서 후리에역변환의 수치구적법

김 희 준

요약: 본 논문에서는 2차원 전기비저항 모델링에서 후리에역변환을 계산하는 수치구적법을 비교하였다. 지수함수 및 큐빅스프라인 보간을 사용한 구적법을 균질대지 모델에 대하여 검토하였다. 이들 기술적용시, λ_{min}을 최소의 샘플링파수라고 할 때 0에서 λ_{min}까지 간격에 대한 적분은 후리에변환된 포텐샬을 대수 함수로 근사함으로써 계산하였다. 이러한 방법은 $\lambda=0$에서의 대수적인 불연속성에 기인한 후리에역변환의 오차를 크게 줄일 수 있다. 수치계산 결과, 샘플링간격이 적당하다면 큐빅스프라인 보간법이 지수함수 보간법보다 더 정확함을 알았다.