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Energy Integrals for Partial Differential Operators of Quasihomogeneous Type

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§0. Introduction

Many authors dealt with the partial differential operators of quasihomogenous type [4], [5], [7], which is not invariant under the change of coordinates in general. In view of singular integral operators Calderon [1] gave the proof on the uniqueness of the Cauchy problem when no bicharacteristic is tangent to the initial surface. Using pseudodifferential operators Lascar [4], [5], studied the propagation of singularities for partial differential operators of quasihomogeneous type. On the other hand Hörmander [2] gave some inequalities from which we can obtain uniqueness theorems for the Cauchy problem, existence and regularity theorems for solutions of the differential equations $P(x, D)u = f$. Isakov [3] extended Hörmander's results to the the equations with quasihomogeneous principal parts.

Let $P(x, D)$ be a partial differential operator defined in an open set $\Omega \subseteq R^n$. Our subjects are related to the following two estimates :

$$(0.1) \qquad \sum_{|\alpha:m| \leq 1-1/M} \int e^{2\tau\omega(x)} |D^\alpha u(x)|^2 \, dx \leqq K_1 \tau^{-1} \int e^{2\tau\omega(x)} |P(x, D)u(x)|^2 \, dx$$

and

$$(0.2) \qquad \sum_{|\alpha:m| \leq 1-1/M} \int e^{2\tau\omega(x)} |D^\alpha u(x)|^2 \, dx \leqq K_2 \tau^{-1} \int e^{2\tau\omega(x)} |P(x, D)u(x)|^2 \, dx + K_3 \sum_{|\alpha:m| \leq 1-2/M} \tau^{2M(1-|\alpha:m|)-2} \int e^{2\tau\omega(x)} |D^\alpha u(x)|^2 \, dx$$

where $u \in C_0^\infty(\Omega)$ and $\tau \geqq \tau_0$. In this note, using commutation relations, we shall find the conditions which are necessary for energy integrals (0.1) or (0.2) to be valid. By means of these conditions Isakov obtained two inequalities which imply the uniqueness theorems of the Cauchy problem for quasihomogeneous partial differential operators.

§ 1. The Preliminaries

For an n-tuple $m = (m_1, \cdots, m_n)$ of positive integers, we write

$$P(x, D) = \sum_{|\alpha:m|\leq 1} a_\alpha(x) D^\alpha \quad \text{with} \quad |\alpha:m| = \sum_{j=1}^{n} \alpha_j/m_j,$$

and define the principal symbol of $P(x, D)$ by $\sigma(P)(x, \xi)$. Then

$$\sigma(P)(x, \xi) = \sum_{|\alpha:m|=1} a_\alpha(x) \xi^\alpha.$$

The variables x_j and ξ_j for which $m_j = M \equiv max_{1\leq k\leq n} m_k$ will be called the fundamental variables. The set of corresponding indices $1 \leq j \leq n$ will be denoted by J, and the complement of J will be denoted by J'. In the sequel we keep

$$\rho = (\rho_1, \cdots, \rho_n), \quad \rho_j = \frac{M}{m_j}, \quad j = 1, \cdots, n,$$

fixed and set

$$A(x) = \sum_{j,k=1}^{n} a_{jk} x_j x_k / 2, \quad a_{jk} = a_{kj},$$
$$\nabla_J \omega(x) = \nabla \omega(x),$$

where $\partial\omega(x) / \partial x_k = 0$, for $k \in J'$.

Definition 1.1. *A complex valued function $f(x, \xi) \in C^\infty$ is said to be ρ-homogeneous of degree $k \in C$ if*

(1.1) $$f_t(x, \xi) \equiv f(x, t^\rho \xi) = t^k f(x, \xi), \quad t > 0$$

where $t^\rho \xi = (t^{\rho_1}\xi_1, \cdots, t^{\rho_n}\xi_n)$.

Definition 1.2. *We say that a partial differential operator $P(x, D)$ is ρ-homogeneous of degree k if the principal symbol of $P(x, D)$ is ρ-homogeneous of degree k*

We shall denote by $[\xi]$ the function defined implicitly by the relation :

(1.2) $$\sum_{j=1}^{n} \frac{|\xi_j|^2}{[\xi]^{2\rho_j}} = 1, \quad \text{if} \quad \xi \in C^n \backslash \{0\},$$
$$[0] = 0$$

We say that g dominates f and write $f << g$ if there exist positive constants C, C' such that

$$f(\xi) \leqq Cg(\xi) \quad \text{for} \quad |\xi| > C'$$

If $f \ll g$ and $g \ll f$, we write $f \sim g$.

Example 1.3. *Let* $M_j = \Pi_{k \neq j}^n \rho_k$ *and* $M = \rho_j M_j$. *Then it follows that*

$$[\xi] \sim \{\sum_{j=1}^n |\xi_j|^{2M_j}\}^{1/2M} \sim \sum_{j=1}^n |\xi_j|^{1/\rho_j}$$

Example 1.4. *For a partial differential operator* $P(x, D) = D_1^3 D_2^2 D_3 + a(x) D_1^2 D_2^2 D_3$, *we take* $m = (12, 4, 4)$. *Then* $\rho = (1, 3, 3)$, $J = \{1\}$ *and* $J' = \{2, 3\}$. *Note that m is not uniquely determined.*

Example 1.5. *For the heat operator* $P(x, t, D_x, D_t) = iD_t + D_1^2 + \cdots + D_n^2 + a(x)D_1$, *we have* $m = (2, \cdots, 2, 1)$, $J = \{1, 2, \cdots, n\}$, $J' = \{n+1\}$ *and* $\rho = (1, \cdots, 1, 2)$.

§ 2. Main Results

We shall prove conditions which are necessary for (0.1) or (0.2) to hold. In doing so we assume that the coefficients of $P(x, D)$ are bounded, that the coefficients in the principal part of $P(x, D)$ are in $C^1(\Omega)$, and that $\omega(x)$ is real valued and belongs to $C^2(\Omega)$.

Theorem 2.1. *Let* $N = \nabla_J \omega(x)$, *and let* $\zeta = \xi + i\gamma N$ *with* $\xi \in R^n$, $\gamma \in R\backslash\{0\}$ *satisfy*

$$\sigma(P)(x, \zeta) = 0. \tag{2.1}$$

Assume that there exists α *with* $|\alpha : m| = 1 - 1/M$ *such that* $\zeta^\alpha \neq 0$. *Then it follows that*

$$\begin{aligned}[\zeta]^{2(M-1)} \leqq 2K_1\{&\sum_{j,k \in J} \omega_{jk}(0)\sigma(P)^{(j)}(0, \zeta)\overline{\sigma(P)^{(k)}(0, \zeta)} \\ &+ \frac{1}{\gamma} Im \sum_{k \in J} \sigma(P)_{(k)}(0, \zeta)\overline{\sigma(P)^{(k)}(0, \zeta)}\},\end{aligned} \tag{2.2}$$

if (0.1) *holds, and that*

$$\begin{aligned}&[\zeta]^{2(M-1)} - K_3\gamma^2[\zeta, \gamma]^{2(M-2)} \\ &\leqq 2K_2\{\sum_{j,k \in J} \omega_{jk}(0)\sigma(P)^{(j)}(0, \zeta)\overline{\sigma(P)^{(k)}(0, \zeta)} \\ &\qquad + \frac{1}{\gamma} Im \sum_{k \in J} \sigma(P)_{(k)}(0, \zeta)\overline{\sigma(P)^{(k)}(0, \zeta)}\},\end{aligned} \tag{2.3}$$

where $[\zeta, \gamma] = \sum_{j=1}^n |\zeta_j|^{\frac{1}{\rho_j}} + |\gamma|$ *if* (0.2) *holds, when the left hand side is positive.*

Theorem 2.2. *Let P have real coefficients and assume that* (2.2) *or* (2.3) *are valid. If* $\nabla_J \omega(x) \neq 0$, $x \in \Omega$, *and if* $\xi \in R^n \backslash \{0\}$ *satisfies*

$$(2.4)\qquad \sigma(P)(x, \xi) = 0, \qquad \sum_{j \in J} \sigma(P)^{(j)}(x, \xi)\frac{\partial \omega}{\partial x_j} = 0,$$

but for some $j \in J$,

$$\sigma(P)^{(j)}(x, \xi) \neq 0.$$

It follows then that

(2.5)

$$[\xi]^{2(M-1)} \leqq C\{\sum_{j,k \in J} \frac{\partial^2 \omega}{\partial x_j \partial x_k}\sigma(P)^{(j)}(x, \xi)\sigma(P)^{(k)}(x, \xi) + \sum_{j,k \in J} \{\sigma(P)^{(k)}_{(j)}(x, \xi)\sigma(P)^{(j)}(x, \xi) - \sigma(P)_{(j)}(x, \xi)\sigma(P)^{(jk)}(x, \xi)\}\frac{\partial \omega}{\partial x_j}\}.$$

Proposition 2.3. *Suppose that* $q(\xi_0) = 0$, $q^{(j)}(\xi_0) \neq 0$ *for some* j,

$$\sum_{j=1}^{n} q^{(j)}(\xi_0) N_j = 0 \qquad \text{for a fixed } N \in C^n.$$

Then there are smooth functions $\xi = \xi(r) \in R^n$ *and* $\tau = \tau(r) \in R^1$ *with* $\xi(0) = \xi_0$ *and* $\tau(0) = 0$ *such that* $q(\xi(r) + i\tau(r)N) = 0$.

Proof: Pick $\eta \in R^n$ so that

$$\sum_{j=1}^{n} q(\xi_0)^{(j)}\eta_j \neq 0.$$

By implicit function theorem, there is an analytic function $z(\tau)$ in the neighborhood of $\tau = 0$ satisfying

$$q(\xi_0 + z\eta + \tau N) = 0.$$

Since $dz/d\tau = 0$ at $\tau = 0$, $z(\tau) \equiv 0$ or $z(\tau) = C\tau^k + O(\tau^{k+1})$ for $k \geqq 2$. There is a smooth curve $\tau = \gamma(r)$ such that $z(\gamma(\tau))$ is real. $\gamma(0) = 0$. Take $\tau = Im\,\gamma(r)$, $\xi(r) = \xi_0 + \eta z(\gamma(r)) + Re\,\gamma(r)N$.

Proof of Theorem 2.2: From the condition (2.4) and from that

$$\sigma(P)^{(k)}(x, \xi + i\gamma N) = \sigma(P)^{(k)}(x, \xi)$$

$$+ i\gamma\{\sum_{j\in T} \sigma(P)^{(kj)}(x, \xi)\frac{\partial\omega}{\partial x_j} + O(\gamma)\}$$

if we take $\gamma = Im\,\gamma(r)$ and replace ξ by $\xi(r) + \eta z(\gamma(r)) + Re\,\gamma(r)N$, both (2.2) and (2.3) are reduced to (2.5), when $\gamma \longrightarrow 0$.

We set

$$\delta_j = D_j + iA_{(j)}(x),$$
$$\overline{\delta_j} = D_j - iA_{(j)}(x),$$
$$\overline{\delta^\alpha} = \prod_{i=1}^{n} \overline{\delta_i^{\alpha_i}}.$$

where $A_{(j)}(x) = \partial A(x)/\partial x_j$. It is obvious then that

$$[\overline{\delta_k}, \delta_j] = 2a_{jk},$$
$$[\overline{\delta_j\delta_k}, \delta_i] = 2a_{ij}\overline{\delta_k} + 2a_{ik}\overline{\delta_j},$$
$$[\overline{\delta_k^l}, \delta_j] = 2la_{kj}\overline{\delta_k^{l-1}},$$

where l is a positive integer. Let $P(D) = \sum_\gamma a_\gamma D^\gamma$ be a partial differential operator with constant coefficients. Since

$$[P(\overline{\delta}), \delta_j] = \sum_\gamma a_\gamma\{[\overline{\delta_1^{\gamma_1}} \cdots \overline{\delta_{n-1}^{\gamma_{n-1}}}, \delta_j]\overline{\delta_n^{\gamma_n}} + \overline{\delta_1^{\gamma_1}} \cdots \overline{\delta_{n-1}^{\gamma_{n-1}}}[\overline{\delta_n^{\gamma_n}}, \delta_j]\},$$

from induction it follows that

$$[\overline{P}(\overline{\delta}), \delta_j] = \sum_{i=1}^{n} 2a_{ij}\frac{\partial \overline{P}(\eta)}{\partial \eta_i}\Big|_{\eta=\overline{\delta}}$$
$$\equiv 2 < A_j, \partial_\eta > \overline{P}(\eta)|_{\eta=\overline{\delta}}$$

where A_j is the $j-th$ column of the matrix (a_{jk}), $\partial_\eta = (\frac{\partial}{\partial_{\eta_1}}, \cdots, \frac{\partial}{\partial_{\eta_n}})$. Next by the direct calculation we obtain

$$[\overline{P}(\overline{\delta}), \delta_j\delta_k] = \{2\delta_k < A_j, \partial_\eta > + 2\delta_j < A_k, \partial_\eta > + 2 < A_j, \partial_\eta > 2 < A_k, \partial_\eta >\}\overline{P}(\eta)|_{\eta=\overline{\delta}}.$$

Repeating this process we can conclude :

Proposition 2.4. *Let $P(D)$, $Q(D)$ be partial differential operators with constant coefficients. It follows then that*

$[\overline{P}(\overline{\delta}), Q(\delta)] =$

$$\sum_{\alpha \neq 0} \frac{1}{\alpha!} Q^{(\alpha)} 2^{|\alpha|} < A_1, \partial^\eta >^{\alpha_1} \cdots < A_n, \partial_\eta >^{\alpha_n} \overline{P}(\eta)|_{\eta = \overline{\delta}}$$

Proposition 2.5. *Let* $H(x) = \sum_{j,k=1}^{n} h_{jk} x_j x_k$, *where* $h_{jk} = h_{kj}$, *be a real duadratic form and let* $b = (b_1, \cdots, b_n)$ *be a vector in* C^n. *Then there exists a constant* $C > 0$ *such that for* $u \in C_0^\infty(R^n)$ *the inequality*

$$\int |u|^2 e^H \, dx \leqq C \int |\sum_{j \in T} b_j D_j u|^2 e^H \, dx$$

is valid if and only if

$$2C \sum_{j,k \in J} h_{jk} b_j \overline{b}_k \geqq 1.$$

Proof: It follows from *Proposition* 2.4 and from *Lemma* 8.1.3 [2].

Proof of Theorem 2.1: We may assume that $x = 0$ and that $\omega(0) = 0$. Consider a function $v \in C^\infty$ such that

$$v^\rho(x) \equiv v(\rho, x) = < x, (\tau/\gamma)^\rho \zeta > + O(|x|^2), \quad x \longrightarrow 0,$$
$$v(x) \equiv v(0, x),$$

and set with a function $\phi \in C_0^\infty(R^n)$

$$u_\tau(x) = e^{iv^\rho(x)} \phi(x\sqrt{\tau}).$$

By Euler identity it is obvious that $\sigma(P)^{(j)}$ is ρ – homogeneous of degree $M(1 - 1/m_j)$. Since

$$e^{-iv^\rho(x)} D_j \{e^{iv^\rho(x)} \phi(x)\} = (\tau/\gamma)^{\rho_j} \partial_j v(x) \phi(x\sqrt{\tau}) + \sqrt{\tau}(D_j\phi)(x\sqrt{\tau}),$$

we obtain

$$P(x, D) u_\tau(x) = (\tau/\gamma)^{M-1} e^{iv^\rho(x)} \{(\tau/\gamma) \sigma(P)(x, \nabla v(x)) \phi(x\sqrt{\tau})$$
$$+ \sum_{j \in T} \sigma(P)^{(j)}(x, \nabla v(x)) \sqrt{\tau} (D_j\phi)(x\sqrt{\tau}) + O(1)\},$$

where $O(1)$ denotes a function which is uniformly bounded for sufficiently large τ. It follows that

$$\tau^{n/2} (\gamma/\tau)^{2(M-1)} \tau^{-1} \int |P(x, D) u_\tau(x)|^2 e^{2\tau\omega(x)} \, dx \longrightarrow$$
$$\int | < x, a > \frac{\phi(x)}{\gamma} + \sum_{j \in T} \sigma(P)^{(j)}(0, \zeta) D_j \phi(x)|^2 e^{2A(x)} \, dx, \quad \tau \longrightarrow \infty.$$

where $a = (D_1 \sigma(P)(x, \nabla v(x)), \cdots, D_n \sigma(P)(x, \nabla v(x)))$. Similarly we find that

$$\tau^{n/2}(\gamma/\tau)^{2(M-1)}\sum_{|\alpha:m|\leq 1-1/M}\int |D^{\alpha}u_{\tau}(x)|^2 e^{2\tau\omega(x)}\,dx$$
$$\longrightarrow \sum_{|\alpha:m|=1-1/M}|\zeta^{2\alpha}| \int |\psi(x)|^2 e^{2A(x)}\,dx$$
$$\approx \int |\psi(x)|^2 e^{2A(x)}\,dx \qquad \tau \longrightarrow \infty,$$

where $2A(x) = \sum_{j,k=1}^{n}\omega_{jk}(x) - Im\, v_{jk}(x)/\gamma$, $g_{jk}(x) = \partial^2 g/\partial x_j\partial x_k$. The last step of the formula above follows from that $|\zeta^{\alpha}| \neq 0$, for some α, $|\alpha:m| = 1 - 1/M$, and that

$$|\zeta^{2\alpha}| = t^{2(M-1)}\prod_j |\zeta_j^2/t^{2\rho_j}|^{\alpha_j}.$$

We therefore obtain the inequality

$$(2.6) \qquad [\zeta]^{2(M-1)} \int |\psi(x)|^2 e^{2A(x)}\,dx$$
$$\leqq K_1 \int | < x, a > \frac{\psi(x)}{\gamma} + \sum_{j\in J} \sigma(P)^{(j)}(0, \zeta)D_j\psi(x)|^2 e^{2A(x)}\,dx,$$

if (0.1) holds. Since $|\zeta^{2\alpha}|\gamma^{2(M-2-<\alpha,\beta>)}$ is $(\rho, 1)$ – homogeneous of degree $2(M-2)$ with respect to (ζ, γ) when $|\alpha:m| = 1 - 2/M$, it follows that

$$\sum_{|\alpha:m|=1-2/M}|\zeta^{2\alpha}|\gamma^{2(M-2-<\alpha,\beta>)} << \{\sum_{j=1}^{n}|\zeta_j|^{\frac{1}{\rho_j}}\} + |\gamma|\}^{2(M-2)}$$
$$\equiv [\zeta, \gamma]^{2(M-2)}.$$

Therefore we obtain from (0.2) the inequality

$$(2.7) \qquad [\zeta]^{2(M-1)} \int |\psi(x)|^2 e^{2A(x)}\,dx$$
$$\leqq K_2 \int | < x, a > \frac{\psi(x)}{\gamma} + \sum_{j\in J} \sigma(P)^{(j)}(0, \zeta)D_j\psi(x)|^2 e^{2A(x)}\,dx$$
$$+ K_3[\zeta, \gamma]^{2(M-2)}\gamma^2 \int |\psi(x)|^2 e^{2A(x)}\,dx.$$

If $\sigma(P)^{(j)}(0, \zeta) = 0$ for all $j \in J$, we immediately find that (2.6) cannot hold by just taking any ψ with support in the neighborhood of the origin where $C_1| < x, a > |^2/\gamma^2 < [\zeta]^{2(M-1)}$. Hence $\sigma(P)^{(j)}(0, \zeta) \neq 0$ for some $j \in J$. Now we choose a function $v(x)$ satisfying the following system of equations

$$(2.8) \qquad \begin{aligned} &\sum_{j\in J} \sigma(P)^{(j)}(0, \zeta)v_{jk}(0) = -\sigma(P)_{(k)}(0, \zeta), \quad k \in J \\ &\sum_{j\in J} \sigma(P)^{(j)}(0, \zeta)v_{jk}(0) = 0, \quad k \in J \\ &\sum_{j\in J} \sigma(P)^{(j)}(0, \zeta)v_{jk}(0) = -\sigma(P)_{(k)}(0, \zeta) \end{aligned}$$

This implies $a = 0$. It is obvious from *Proposition* 2.5 that

$$(2.9)\qquad [\zeta]^{2(M-1)} \leqq 2K_1 \sum_{j,k\in J} \{\omega_{jk}(0) - \frac{Im\, v_{jk}(0)}{\gamma}\} \times \sigma(P)^{(j)}(0,\zeta)\overline{\sigma(P)^{(k)}(0,\zeta)}.$$

Now we have, in view of (2.8)

$$\sum_{j,k\in J} v_{jk}(0)\sigma(P)^{(j)}(0,\zeta)\overline{\sigma(P)^{(k)}(0,\zeta)} = -\sum_{k\in J}(P)_{(k)}(0,\zeta)\overline{\sigma(P)^{(k)}(0,\zeta)}.$$

Thus we obtain

$$[\zeta]^{2(M-1)} \leqq 2K_1\{\sum_{j,k\in J}\omega_{jk}(0)\sigma(P)^{(j)}(0,\zeta)\overline{\sigma(P)^{(k)}(0,\zeta)}. + \frac{1}{\gamma} Im\sum_{k\in J}\sigma(P)_{(k)}(0,\zeta)\overline{\sigma(P)^{(k)}(0,\zeta)}\},$$

Next we assume that (2.7) holds. When $[\zeta]^{2(M-1)} < C_3[\zeta,\gamma]^{2(M-2)}\gamma^2$, it follows as before that $\sigma(P)^{(j)}(0,\zeta) \neq 0$ for some $j \in J$. Repeating argument above, we again obtain (2.3).

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