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Inertial Manifold for Finite Element Approximation of Reaction Diffusion Equation*

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1. Introduction

Let Ω be a square and consider the scalar partial differential equations of the form

$$u_t = \Delta u + f(x, u) \quad \text{in } \Omega \tag{1.1}$$

with homogeneous Dirichlet boundary condition, i.e., $u = 0$ on the boundary of Ω. We shall assume that the nonlinear term $f : \Omega \times R \to R$ is smooth and satisfies the following conditions for some positive constants K_1 and K_2 ,

$$\begin{aligned} &|f(x, u)|, \quad |D_x f(x, u)| \leq K_1|u| + K_2 \quad \text{and} \\ &|D_u f(x, u)| \leq K_1 \quad \text{in } \bar{\Omega} \times R . \end{aligned} \tag{1.2}$$

Under these assumptions one sees that the equation (1.1) admits an inertial manifold. See, for example, Foias, Sell and Temam (1988) for reference.

In this paper we shall consider the "continuous time" discretization of (1.1) using a second order Galerkin method and we prove that for sufficiently small discretization parameter, the approximate equation has the same dimensional inertial manifold which converges to that of (1.1) in the certain operator norm (see Theorem 2.1).

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2. Abstract Setting and Main Results

Let $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$. We introduce a continuous, symmetric, bilinear form $a(\cdot, \cdot)$ on $V \times V$ defined by

$$a(u, v) = \int \nabla u \cdot \nabla v d\Omega$$

and the corresponding operator A on V with domain $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$. The bilinear form is coercive, i.e., there is a positive constant K_3 such that

$$a(u, u) \geqq K_3 |u|_V^2$$

and the operator A is sectorial and generates and analytic semigroup. Moreover we have $\mathcal{D}(A^{1/2}) = H_0^1(\Omega)$. (See, for example, Henry (1981).)

It is clear from (1.2) that f induces a map $f: V \to V$ defined by

$$[f(u)](x) = f(x, u(x)), \quad u \in V$$

and we can now write the equation (1.1) as an abstract differential equation

$$\frac{du}{dt} + Au = f(u), \quad u(0) = u_0 \tag{2.1}$$

where u_0 is an initial condition in V.

One sees that the initial value problem (2.1) possesses an unique solution and the existence of global solution and global attractor, $\mathcal{A}$, is guaranteed by virtue of the growth conditions on f. For details, we refer to Temam (1988).

Now let us turn to a finite dimensional approximation of the equation (2.1). For concreteness, we consider the "continuous time" discretization of (2.1) using a second order Galerkin method. For a sequence of discretization parameter $h \in (0, \frac{1}{2}]$ tending to zero, let T_h be a regular family of triangulations in the sense of Cialet (1978) where T_h is made of triangles with diameters bounded by h and we set

$$V_h = \{v_h \in C^0(\bar{\Omega}) \cap H_0^1(\Omega) \mid \forall K \in T_h, v_h|_K \in P_1(K)\}, \tag{2.2}$$

where $P_1(K)$ is the space of all polynomials of degree $\leqq 1$ on K. Then V_h is a finite dimensional subspaces of $H_0^1(\Omega)$.

From now on we shall denote by $|\cdot|_H$, $|\cdot|_V$, $(\cdot, \cdot)_H$ and $(\cdot, \cdot)_V$, the norms and inner products of the spaces H, V respectively and we introduce the operator $A_h \in \mathcal{L}(V_h ; V_h)$ defined by

$$\forall v_h \in V_h, \quad (A_h w_h, v_h)_H = a(w_h, v_h) \quad \text{for } w_h \in V_h. \tag{2.3}$$

Then A_h is a self-adjoint, positive definite, continuous, linear operator on V_h.

Let $L_h : H \to V_h$ be the orthogonal projection on V_h (usually called L^2-projection) i.e.

$$(v - L_h v, v_h)_H = 0 \quad \forall v \in H, v_h \in V_h$$

and let $R_h : V \to V_h$ be the orthogonal projection on V_h (usually called elliptic projection or Ritz projection) i.e.

$$a(v - R_h v, v_h) = 0 \quad \forall v \in V, v_h \in V_h.$$

These are standard orthogonal projections on closed subspaces of Hilbert spaces.

Then we have the approximate equation on V_h

$$\frac{du_h}{dt} + A_h u_h = L_h f(u_h), \quad u_h(0) = u_{0h}, \tag{2.4}$$

where $u_{0h} \in V_h$. This equation is an ordinary differential equation and because of our assumptions about nonlinear term f, the solution $u_h(t)$ exists for all positive t.

Under the following hypothesis on the spaces $\{V_h\}$; there exist an integer $m > 0$ and, for any β, $\frac{1}{2} \leqq \beta \leqq 1$, a constant $C(\beta) > 0$ such that, for all $w \in X^\beta \equiv \mathcal{D}(A^\beta)$,

$$\begin{aligned} |w - R_h w|_V + |w - L_h w|_V &\leqq C(\beta) h^{2m(\beta - \frac{1}{2})} |w|_{X^\beta} \\ |w - R_h w|_H + |w - L_h w|_H &\leqq C(\beta) h^{2m\beta} |w|_{X^\beta} \end{aligned} \tag{2.5}$$

Hale, Lin and Raugel (1988) proved that the approximate equation (2.4) also has a global attractor $\mathcal{A}_h$ and it is upper-semicontinuous at $h = 0$, that is

$$\delta_V(\mathcal{A}_h, \mathcal{A}) = \sup_{u \in \mathcal{A}_h} \inf_{v \in \mathcal{A}} |u - v|_V \to 0 \quad \text{as } h \to 0. \tag{2.6}$$

They also remarked that the second order Galerkin method gives the estimates (2.5). For details, see Hale, Lin and Raugel (1988) and the references cited there. Here m is related to the order of differential operator. For the Laplace operator we have $m = 1$.

Our main goal is to compare the dynamics of the attractors, so we introduced a standard smooth cut-off function $\phi(x) : R \to [0, 1]$ such that $\phi(x) = 0$ for $x \geqq \rho_0^2$ and $\phi(x) = 1$ for $x \leqq \frac{\rho_0^2}{2}$ where ρ_0 is chosen so that

$$(2.7) \qquad \mathcal{A} \subset \{v \in V : |v|_V \leqq \frac{\rho_0}{2}\}$$

$$\mathcal{A}_h \subset \{v \in V : |v|_V \leqq \frac{\rho_0}{2}\} \cap V_h \quad \text{for } h > 0$$

and we modify $f(u)$ by $\phi(|u|_V^2)f(u) \overset{\text{def}}{=} F(u)$. This type of modification is standard and from now on we assume that the modification has been made. For $h > 0$, $F_h(u)$ is defined to be $L_h \phi(|u_h|_V^2)f(u_h)$.

Hence we obtain new equations

$$(2.8) \qquad \frac{du}{dt} + Au = F(u), \quad u(0) = u_0 \text{ on } V,$$

$$(2.9) \qquad \frac{du_h}{dt} + A_h u_h = F_h(u_h), \quad u_h(0) = u_{0h} \text{ on } V_h.$$

Let $0 < \lambda_1 \leqq \lambda_2 \leqq \cdots$ and $0 < \lambda_{1,h} \leqq \lambda_{2,h} \leqq \cdots$ denote the eigenvalues of A and A_h respectively repeated with their multipicities and let $\phi_1, \phi_2, \cdots$ and $\phi_{1,h}, \phi_{2,h}, \cdots$ denote the corresponding eigenvectors of A and A_h. We assume that the eigenvectors form an orthonormal set in H. For $N \geqq 1$, we let $P = P_N$, $P_h = P_{N,h}$ denote the orthogonal projection of H onto span$\{\phi_1, \phi_2, \cdots, \phi_N\}$, span$\{\phi_{1,h}, \phi_{2,h}, \cdots, \phi_{N,h}\}$ respectively and $Q = I - P$, $Q_h = I - P_h$.

We will construct the inertial manifold $\mathcal{M}$, $\mathcal{M}_h$ for (2.8), (2.9) as the graph of functions $\Phi : PV \to QV$, $\Phi_h : P_hV_h \to Q_hV_h$ and before we state our main theorem we introduce the natural isomorphisms

$$\pi_h : P_hV_h \to R^N \quad \text{for} \quad h \geqq 0$$

which is chosen so that $|p|_V = |\pi_h p|_{R^N}$. Here $|\cdot|_{R^N}$ denote the Euclidean norm of R^N and we take $P_0 = P$, $V_0 = V$. From now on we also take $\pi_0 = \pi$, $\Phi_0 = \Phi$. Let us look at the diagram

$$R^N \xrightarrow{\pi_h^{-1}} P_hV_h \xrightarrow{\Phi_h} Q_hV_h$$

$$R^N \xrightarrow{\pi^{-1}} PV \xrightarrow{\Phi} QV$$

We compare Φ_h via the projection π_h. We define $\Phi_h^*(\alpha) = \Phi_h(\pi_0^{-1}(\alpha))$ for $h \geqq 0$ and

$$\|\Phi_h^* - \Phi_0^*\|_V = \sup_{\alpha \in R^N} |\Phi_h(\pi_h^{-1}(\alpha)) - \Phi_0(\pi_0^{-1}(\alpha))|_V$$

and the derivative of Φ_h^* as a function from R^N to V is simply

$$D\Phi_h^*(\alpha)(\beta) = D\Phi_h(\pi_h^{-1}(\alpha))\pi_h^{-1}(\beta)$$

hence we can define

$$\|D\Phi_h^* - D\Phi_0^*\|_V = \sup_{\alpha,\beta \in R^N} |D\Phi_h(\pi_h^{-1}(\alpha))\pi_h^{-1}(\beta) - D\Phi_0(\pi_0^{-1}(\alpha))\pi_0^{-1}(\beta)|_V .$$

Here D is the derivative of Φ_h as a function from P_hV_h to Q_hV_h and the sup is taken over α, β with $|\beta|_{R^N} = 1$. Since there can be no confusion we write $\|D\Phi_h^* - D\Phi_0^*\|_V = \|D\Phi_h - D\Phi_0\|_V$. Now we state our main Theorems and we shall provide their proofs in the next two sections.

Theorem 2.1. *For some $N > 0$, there exist N-dimensional inertial manifolds $\{\mathcal{M}_h\}$ of (2.8) and (2.9) such that $\mathcal{M}_h$ are the graph of the smooth functions Φ_h which satisfy*

$$\|\Phi_h - \Phi_0\|_V \leqq \delta_1(h) \quad \text{and} \tag{2.10}$$

$$\|D\Phi_h - D\Phi_0\|_V \leqq \delta_2(h) \tag{2.11}$$

where $\delta_1(h)$, $\delta_2(h) \to 0$ as $h \to 0$.

In fact, Φ_h is constructed as the fixed point of the Lyapunov–Perron operator, i.e.,

$$\Phi_h(p_{0h}) = \int_{-\infty}^{0} e^{A_hQ_hs}Q_hF_h(p_h(s) + \Phi_h(p_h(s)))ds \tag{2.12}$$

where $p_h(t)$ is the solution of the inertial form

$$\frac{dp}{dt} + A_hP_hp = P_hF_h(p + \Phi_h(p))$$

that satisfies $p_h(0) = p_{0h}$.

As a result of Theorem 2.1 , we have

Theorem 2.2. *The flow on the inertial manifold $\mathcal{M}_h$ is determined by an N-dimensional ordinary differential equation*

$$\frac{dp}{dt} = G_h(p),\ p \in R^N \tag{2.13}$$

and the function G_h converges to G_0 in the space $C^1(B_{\rho_0}, R^N)$ as $h \to 0$, where B_{ρ_0} is a ball in R^N of radius ρ_0.

Remark. It is remarkable that Theorem 2.1 and Theorem 2.2 imply that the hyperbolic structure is preserved by the finite element approximation. For this kind of result, see Pliss and Sell (1990). Precise and even general results will be discussed subsequently.

3. Basic Estimates

First, one easily sees that the nonlinear term F satisfies the "Standing Hypothesis" as given in Luskin and Sell (1989), i.e., there are positive constants C_0 and C_1 depending only on K_1, K_2, K_3 and ρ_0 such that $F: V \to V$ has a continuous Gateaux derivative and satisfies

$$(3.1)\qquad \begin{aligned} |F(u)|_V &\leqq C_0 \quad \text{for all } u \in V \\ |D_uF(u)v|_V &\leqq C_1|v|_V \quad \text{for all } u,\ v \in V \end{aligned}$$

where $D_uF(u)$ denotes the Gateaux derivative of F at the point u and furthermore

$$(3.2)\qquad \operatorname{supp} F \subset \{u \in V : |v|_V \leqq \rho_0\}.$$

It is also easy to check the "Standing Hypothesis" for the nonlinear term F_h. In fact $F_h(u)$ has bounded support in the ball $\{u_h \in V_h : |u_h|_V \leqq \rho_0\}$ and

$$\begin{aligned} |F_h(u)|_V &= |L_h \phi(|u_h|_V^2) f(u_h)|_V \\ &\leqq |(L_h - I)\phi(|u_h|_V^2) f(u_h)|_V + |\phi(|u_h|_V^2) f(u_h)|_V \\ &\leqq (C(\beta = \tfrac{1}{2}) + 1)|\phi(|u_h|_V^2) f(u_h)|_V \quad \text{be (2.5)} \\ &\leqq (C(\beta = \tfrac{1}{2}) + 1) \sup_{|u|_V^2 \leqq \rho_0} |f(u)|_V \\ &\leqq \widetilde{C}_0. \end{aligned}$$

Similarly

$$D_uF_h(u_h)v_h = L_h \phi_x(|u_h|_V^2)\ a_h(u_h, v_h) f(u_h) + L_h \phi(|u_h|_V^2) D_u f(u_h) v_h$$

and

$$\begin{aligned} |D_uF_h(u_h)v_h|_V &\leqq (C(\beta = \tfrac{1}{2}) + 1)\left(\sup_{|u|_V^2 \leqq \rho_0} |f(u)|_V + \sup_{|u|_V^2 \leqq \rho_0} |D_u f(u)|_V\right)|v_h|_V \\ &\leqq \widetilde{C}_1 |v_h|_V. \end{aligned}$$

We may assume $C_0 = \widetilde{C}_0$ and $C_1 = \widetilde{C}_1$.

The existence theory in Luskin and Sell (1989) tells us that there are constants M_0 and M_1, depending only on C_1 such that if the eigenvalues of A_h satisfy $\lambda_{N+1,h} \geqq M_0$ together with the gap condition $\lambda_{N+1,h} - \lambda_{N,h} \geqq M_1$, then (2.8) and (2.9) have inertial manifolds of the form $\mathcal{M}_h = \text{graph}\,\Phi_h$ where

(1) $\Phi_h : P_h V_h \to Q_h V_h$ is a smooth function with $\text{supp}\,\varphi_h \subset \{p_h \in P_h V_h : |p_h|_V \leqq \rho_0\}$

(2) $|D\varphi_h(p)|_V \leqq 1$ for all $p \in P_h V_h$ and $h \geqq 0$.

Actually the operator A satisfies the spectral gap condition for some N and we keep this N. Note that the eigenbalues of A have the form $(m_1^2 + m_2^2)$ where m_1 and m_2 are integers. In this case, a result from number theory implies that the spectral gap condition is satisfied, cf. Richards (1982). We prepare several estimates for the proof of our main Theorems. First of all we have

$$\sup_{n \leqq N+1} |\lambda_{n,h} - \lambda_n| \leqq \eta_1(h) \text{ where } \eta_1(h) \to 0 \text{ as } h \to 0, \tag{3.3}$$

$$\sup_{n \leqq N} |\phi_{n,h} - \phi_n|_V \leqq \eta_2(h) \text{ where } \eta_2(h) \to 0 \text{ as } h \to 0. \tag{3.4}$$

These are proved in Strang and Fix by using Rayleigh quotient and the Minimax principle for the $2m$-order elliptic operator. We state below ;

Theorem 3.1. *V_h is such that $|u - R_h u|_{H^s} \leqq Ch^{r+1-s}|u|_{H^{r+1}}$, then for small h and for $n \leqq \dim V_h$*

$$\lambda_n \leqq \lambda_{n,h} \leqq \lambda_n + 2Ch^{2(r+1-m)}\lambda_n^{\frac{r+1}{m}}. \tag{3.5}$$

Moreover the orthonormal eigenfuntions ϕ_n can be chosen so that the following estimate holds

$$|\phi_n - \phi_{n,h}|_V \leqq Ch^{2(r+1-m)}\lambda_n^{\frac{r+1}{m}}. \tag{3.6}$$

In particular, note that (3.3) guarantees not only the existence of inertial manifolds of (2.9) for $h > 0$ but also that they have the same dimension N for $0 \leqq h \leqq h_0$ for some h_0 simply because $\lambda_{N+1,h} - \lambda_{N,h} \geqq \lambda_{N+1} - \lambda_N - 2\eta_1(h)$ and $\eta_1(h) \to 0$ also note that (3.4) gives the estimate

$$|(u, \phi_n)_H \phi_n - (u, \phi_{n,h})_H \phi_{n,h}|_V \leqq \eta_2(h)|u|_H \quad \text{for } n \leqq N. \tag{3.7}$$

We provide other estimates which are needed to compare $\{\Phi_h\}$:

$$|(e^{-At} - e^{-A_h t})u_0|_V \leqq \eta_3(h, t)|u_0|_V \text{ for } u_0 \in V_h,\ t > 0, \tag{3.8}$$

where for any $T > 0$, $\int_0^T \eta_3(h, t)dt \to 0$ as $h \to 0$.

$$(3.9)\quad \begin{aligned} |F(v) - F_h(v)|_H &\leq \eta_4(h) \quad \text{for all } v \in V_h, \\ |D_uF(v)w - D_uF_h(v)w|_H &\leq \eta_5(h)|w|_V \quad \text{for all } v,\ w \in V_h, \end{aligned}$$

where $\eta_2, \eta_4, \eta_5 \to 0$ as $h \to 0$.

The estimates (3.9) are easily obtained by using (2.5), namely,

$$\begin{aligned} |F(u) - F_h(u)|_H &= |\phi(|u|_V^2)f(u) - L_h\phi(|u|_V^2)f(u))|_H \\ &\leq Ch^m|\phi(|u|_V^2)f(u)|_V \\ &\leq Ch^m \quad \text{for } u \in V_h. \end{aligned}$$

$$\begin{aligned} |D_uF(u)w - D_uF_h(u)w|_H &= |D_u\{\phi(|u|_V^2f(u)\}w - L_hD_u\{\phi(|u|_V^2)f(u)\}w|_H \\ &\leq Ch^m|D_u\{\phi(|u|_V^2)f(u)\}w|_V \\ &\leq Ch^m|w|_V \quad \text{for } u,\ w \in V_h. \end{aligned}$$

The estimates (3.8) is proved in the following theorem

Theorem 3.2. *Let A be a strongly elliptic self-adjoint operator of order 2m with homogeneous boundary condition. If u is a solution in V of the equation*

$$\frac{du}{dt} + Au = 0, \quad u(0) = u_0$$

and u_h is a solution of the approximate equation

$$\frac{du_h}{dt} + A_hu_h = 0, \quad u_h(0) = u_0$$

then we have

$$(3.10)\quad |u - u_h|_V \leq t^{-\frac{1}{2}}h^m|u_0|_V \quad \textit{for} \quad u_0 \in V_h \quad \textit{and}\ t > 0.$$

Proof. This can be proved by using energy techniques combined with parabolic duality arguments. This and more general results can be found in Luskin and Rannacher (1981) or Thomee (1984).

4. Proof of the Main Theorem

Since Theorem 2.2 is a direct consequence of Theorem 2.1, we mainly prove Theorem 2.1. In particular we focus our attention only to the convergence of $\{\Phi_h\}$.

4.1 C^0 convergence of $\{\Phi_h\}$.

We recall that for $0 \leqq h \leqq h_0$, Φ_h is the fixed point of the Lyapunov Perron operator, i.e.,

$$\Phi_h(p_{0h}) = \int_{-\infty}^{0} e^{A_hQ_hs}Q_hF_h(p_h(s) + \Phi_h(p_h(s)))ds \tag{4.1}$$

where $p_h(t)$ is the solution of the inertial form

$$p_{h,t} + A_hP_hp_h = P_hF_h(p_h + \Phi_h(p_h)) \tag{4.2}$$

that satisfies $p_h(0) = p_{0h}$.

Since $\{\Phi_h\}$ are sitting in different spaces, we must be more careful of comparing them. We use the natural isormorphism $\pi_h : P_hV_h \to R^N$ and our goal is to get the estimates of

(4.3)

$$\begin{aligned}
&\Phi_h(\pi_h^{-1}(\alpha)) - \Phi_0(\pi_0^{-1}(\alpha)) = \Phi_h(p_h) - \Phi_0(p_0) \\
&= \int_{-\infty}^{0} e^{A_hQ_hs}Q_hF_h(p_h(s) + \Phi_h(p_h(s)))ds - \int_{-\infty}^{0} e^{AQs}QF(p(s) + \Phi(p(s)))ds \\
&= \int_{-T}^{0} (e^{A_hQ_hs}Q_h - e^{AQs}Q)F_h(p_h(s) + \Phi_h(p_h(s)))ds \\
&\quad + \int_{-T}^{0} e^{AQs}Q[F_h(p_h(s) + \Phi_h(p_h(s))) - F(p(s) + \Phi(p_h(s)))]ds \\
&\quad + \int_{-\infty}^{-T} e^{A_hQ_hs}Q_hF_h(p_h(s) + \Phi_h(p_h(s)))ds - \int_{-\infty}^{-T} e^{AQs}QF(p_h(s) + \Phi(p(s)))ds
\end{aligned}$$

for $|\alpha|_{R^N} \leqq \rho_0$.

For the last two terms, one has

$$\begin{aligned}
\left| \int_{-\infty}^{-T} e^{A_hQ_hs}Q_hF_h(p_h(s) + \Phi_h(p_h(s)))ds \right|_V &\leqq C_0 \int_{-\infty}^{-T} e^{\lambda_{N+1,h}s}ds \\
&= \frac{C_0}{\lambda_{N+1,h}} e^{-\lambda_{N+1,h}T},
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
\left| \int_{-\infty}^{-T} e^{AQs}QF(p_h(s) + \Phi(p_h(s)))ds \right|_V &\leqq C_0 \int_{-\infty}^{-T} e^{\lambda_{N+1}s}ds \\
&= \frac{C_0}{\lambda_{N+1}} e^{-\lambda_{N+1}T}.
\end{aligned} \tag{4.5}$$

Before we proceed, we prepare the following lemmas.

Lemma 4.1.

$$|\pi_h^{-1}(\pi(p)) - p|_V \leqq N\eta_2(h)|p|_H \quad \textit{for} \quad p \in PH.$$

Proof. Let $p = \sum_{n\leq N} C_n \phi_n$, then $\pi_h^{-1}(\pi(p)) = \sum_{n\leq N} C_{n,h}$ and

$$|\pi_h^{-1}(\pi(p)) - p|_V \leq \sum_{n\leq N} |c_n||\phi_n - \phi_{n,h}|_V \leq N\eta_2(h)|p|_H \quad \text{by (3.4)}.$$

Lemma 4.2. *Let* $\lambda_* = \min(\lambda_1, \lambda_{1,h})$ *and* $\lambda^* = \max(\lambda_N, \lambda_{N,h})$ *for* $0 \leq h \leq h_0$. *For* $u \in V_h$, *we have*

$$|(e^{tA_hP_h}P_h - e^{tAP}P)u|_V \leq \begin{cases} e^{\lambda_* t}\eta_6(h)|u|_H, & \text{for } t < 0 \\ e^{\lambda^* t}\eta_7(h)|u|_H, & \text{for } t > 0. \end{cases}$$

where $\eta_6(h), \eta_7(h) \to 0$ *as* $h \to 0$.

Proof. Let $u = \sum c_n \phi_n = \sum c_{n,h}\phi_{n,h}$, then

$$(e^{tA_hP_h}P_h - e^{tAP}P)u = \sum_{n\leq N} (e^{t\lambda_{n,h}}c_{n,h}\phi_{n,h} - e^{t\lambda_n}c_n\phi_n).$$

For $t < 0$,

$$\begin{aligned} &|(e^{tA_hP_h}P_h - e^{tAP}P)u|_V \\ &\leq e^{\lambda_* t}|\sum_{n\leq N} e^{t(\lambda_{n,h}-\lambda_*)}(c_{n,h}\phi_{n,h} - c_n\phi_n) + \sum_{n\leq N} (e^{t(\lambda_{n,h}-\lambda_*)} - e^{t(\lambda_n-\lambda_*)})c_n\phi_n|_V \\ &\leq e^{\lambda_* t}\{N\eta_2(h)|u|_H + \sum_{n\leq N} |e^{t(\lambda_{n,h}-\lambda_*)} - e^{t(\lambda_n-\lambda_*)}||\phi_n|_V|u|_H\} \quad \text{by (3.7)} \\ &\leq e^{\lambda_* t}\eta_6(h)|u|_H, \end{aligned}$$

where

$$\eta_6(h) = N\eta_2(h) + \sup_{t<0}\sum_{n\leq N} |e^{t(\lambda_{n,h}-\lambda_*)} - e^{t(\lambda_n-\lambda_*)}||\phi_n|_V. \tag{4.6}$$

On the other hans, for $t > 0$,

$$\begin{aligned} &|(e^{tA_hP_h}P_h - e^{tAP}P)u|_V \\ &\leq e^{\lambda^* t}|\sum_{n\leq N} e^{t(\lambda_{n,h}-\lambda^*)}(c_{n,h}\phi_{n,h} - c_n\phi_n) + \sum_{n\leq N} (e^{t(\lambda_{n,h}-\lambda^*)} - e^{t(\lambda_n-\lambda^*)})c_n\phi_n|_V \\ &\leq e^{\lambda^* t}\eta_7(h)|u|_H \end{aligned}$$

where

$$\eta_7(h) = N\eta_2(h) \ \sup_{t>0}\sum_{n\leq N} |e^{t(\lambda_{n,h}-\lambda^*)} - e^{t(\lambda_n-\lambda^*)}||\phi_n|_V. \tag{4.7}$$

In particular $\eta_6(h), \eta_7(h) \to 0$ as $h \to 0$ because of (3.3).

Lemma 4.3. *Let $p_h(t)$ be the solution of*

$$(4.8)\qquad \frac{dp_h}{dt} + A_hP_hp_h = P_hF_h(p_h + \Phi_h(p_h))$$

that satisfies the initial condition $p_h(0) = p_{h,0}$ with $|p_{h,0}|_V \leqq \rho_0$ and $p(t)$ be the solution of

$$(4.9)\qquad \frac{dp}{dt} + APp = PF(p + \Phi(p))$$

that satisfies the initial condition $p(0) = p_0$ with $|p_0|_V \leqq \rho_0$, then for $h > 0$

$$|p_h(t) - p(t)|_V \leqq e^{-t\lambda^*}\eta_{10}(h) + \frac{2C_1}{\lambda_N} e^{-t\lambda_N}\|\Phi - \Phi_h\|_V$$

where $t < 0$ and $\eta_{10}(h) \to 0$ as $h \to 0$.

Proof. From the variation of constant formula, we have

$$p_h(t) = e^{-tA_hP_h}p_h(0) - \int_t^0 e^{(s-t)A_hP_h}P_hF_h(p_h + \Phi_h(p_h))ds .$$

Hence

$$\begin{aligned} p_h(t) - p(t) = {} & (e^{-tA_hP_h} - e^{-tAP})p_h(0) + e^{-tAP}(p_h(0) - p(0)) \\ & - \int_t^0 (e^{(s-t)A_hP_h}P_h - e^{(s-t)AP}P)F_h(p_h + \Phi_h(p_h))ds \\ & - \int_t^0 e^{(s-t)AP}P(F_h(p_h + \Phi_h(p_h)) - F(p_h + \Phi_h(p_h)))ds \\ & - \int_t^0 e^{(s-t)AP}P(F(p_h + \Phi_h(p_h)) - F(p + \Phi(p)))ds \end{aligned}$$

and

(4.10)

$$\begin{aligned} |p_h(t) - p(t)|_V \leqq {} & e^{-t\lambda^*}\eta_7(h)|p_h(0)|_H + e^{-t\lambda_N}|p_h(0) - p(0)|_V \\ & + C_0\int_t^0 e^{(s-t)\lambda^*}\eta_7(h)ds \quad \text{(using Lemma 4.2)} \\ & + \int_t^0 \sqrt{\lambda_N}\, e^{(s-t)\lambda_N}\eta_4(h)ds \quad \text{by (3.9)} \\ & + \int_t^0 e^{(s-t)\lambda_N}C_1[|p_h(s) - p(s)|_V + |\Phi_h(p_h(s)) - \Phi(p(s))|_V]ds. \end{aligned}$$

Now

$$\begin{aligned} |\Phi_h(p_h(s)) - \Phi(p(s))|_V \leqq {} & |\Phi_h(p_h(s)) - \Phi_h(\pi_h^{-1}(\pi(p(s))))|_V \\ & + |\Phi_h(\pi_h^{-1}(\pi(p(s)))) - \Phi(p(s))|_V \end{aligned}$$

$$\leqq |p_h(s) - \pi_h^{-1}(\pi(p(s)))|_V + ||\Phi - \Phi_h||_V$$
$$\leqq |p_h(s) - p(s)|_V + |p(s) - \pi_h^{-1}(\pi(p(s)))|_V + ||\Phi - \Phi_h||_V.$$

Applying Lemma 4.1 , we obtain

(4.11) $$|\Phi_h(p_h(s)) - \Phi(p(s))|_V \leqq |p_h(s) - p(s)|_V + ||\Phi - \Phi_h||_V + N\eta_2(h)|p(s)|_H .$$

In order to estimate $|p(s)|_H$, we take the scalar product of (4.9) with $p(s)$ and get

$$\frac{1}{2}\frac{d}{dt}|p|_H^2 + |A^{\frac{1}{2}}p|_H^2 \geqq -|PF(p + \Phi(p))|_H|p|_H$$

and

$$\frac{d}{dt}|p|_H \geqq -\lambda_N|p|_H - C_0.$$

Hence we obtain

(4.12) $$|p(s)|_H \leqq (|p(0)|_H + \frac{C_0}{\lambda_N})e^{-\lambda_N s} \quad \text{for } s < 0 .$$

By using (4.10), (4.11) and (4.12) we get

$$\begin{aligned} e^{t\lambda_N}|p_h(t) - p(t)|_V \leqq\ & e^{-(\lambda^*-\lambda_N)t}\frac{\rho_0}{\lambda_1^{\frac{1}{2}}}\eta_7(h) + N\ \rho_0\eta_2(h) \\ & + \frac{C_0\eta_7(h)}{\lambda^*} \quad (e^{-t(\lambda^*-\lambda_N)} - e^{t\lambda_N}) + \frac{1}{\lambda_N^{\frac{1}{2}}}(1 - e^{\lambda_N t})\eta_4(h) \\ & + 2C_1 \int_t^0 e^{s\lambda_N}|p_h(s) - p(s)|_V ds \\ & + C_1 \int_t^0 e^{s\lambda_N}(\frac{\rho_0}{\lambda_1^{\frac{1}{2}}} + \frac{C_0}{\lambda_N})N\eta_2(h)e^{-\lambda_N s}ds \\ & + C_1 \int_t^0 e^{s\lambda_N}ds||\Phi - \Phi_h||_V \\ \leqq\ & e^{-(\lambda-\lambda_N)t}\eta_8(h) - t\eta_9(h) + \frac{C_1}{\eta_N}||\Phi - \Phi_h||_V \\ & + 2C_1 \int_t^0 e^{s\lambda_N}|p_h(s) - p(s)|_V ds, \end{aligned}$$

where $\eta_8(h)$, $\eta_9(h) \to 0$ as $h \to 0$.

By the Gronwall inequality, one gets

$$\begin{aligned} & \int_t^0 e^{s\lambda_N}|p_h(s) - p(s)|_V ds \\ & \qquad \leqq \int_t^0 e^{2C_1 s}\{e^{-(\lambda^*-\lambda_N)s}\eta_8(h) - s\eta_9(h) + \frac{C_1}{\lambda_N}||\Phi - \Phi_h||_V\}ds \end{aligned}$$

$$\leq \frac{\eta_8(h)}{2C_1-\lambda^*+\lambda_N} + \frac{\eta_9(h)}{4C_1^2} + \frac{1}{2\lambda_N}\|\Phi - \Phi_h\|_V.$$

Therefore

$$e^{t\lambda_N}|p_h(t) - p(t)|_V \leq e^{-(\lambda^*-\lambda_N)t}\eta_8(h) - t\eta_9(h) + \frac{C_1}{\lambda_N}\|\Phi - \Phi_h\|_V$$
$$+ 2C_1\{\frac{\eta_8(h)}{2C_1-\lambda^*+\lambda_N} + \frac{\eta_9(h)}{4C_1^2} + \frac{1}{2\lambda_N}\|\Phi - \Phi_N\|_V\}$$

and

$$|p_h(t) - p(t)|_V \leq e^{-t\lambda^*}\eta_{10}(h) + \frac{2C_1}{\lambda_N} e^{-t\lambda_N}\|\Phi - \Phi_h\|_V,$$

where $\eta_{10}(h) \to 0$ as $h \to 0$. This completes the proof of Lemma 4.3.

Let us return to the equation (4.3). Since

$$(e^{A_h s} - e^{As})u = (e^{A_h Q_h s}Q_h - e^{AQs}Q)u + (e^{A_h P_h s}P_h - e^{APs}P)u \quad \text{for } u \in V_h ,$$

using (3.8) and Lemma 4.2, after applying $A^{\frac{1}{2}}$, the first term on the right of the equation (4.3) is bounded by

$$\int_{-T}^{0}|A^{\frac{1}{2}}(e^{A_h Q_h s}Q_h - e^{AQs}Q)F_h(p_h(s) + \Phi_h(p_h(s)))|_V ds$$
$$\leq C_0 \int_{-T}^{0}(\eta_3(h, s) + e^{\lambda_* s}\eta_6(h))ds$$
$$= \eta_{11}(h, T).$$

Note that $\eta_{11}(h, T) \to 0$ as $h \to 0$ for any fixed $T > 0$.

After applying $A^{\frac{1}{2}}$ to the equation (4.3), the second term on the right becomes

$$\int_{-T}^{0}A^{\frac{1}{2}}e^{AQs}Q[F_h(p_h(s) + \Phi_h(p_h(s))) - F(p_h(s) + \Phi_h(p_h(s)))]ds$$
$$+ \int_{-T}^{0}A^{\frac{1}{2}}e^{AQs}Q[F(p_h(s) + \Phi_h(p_h(s))) - F(p(s) + \Phi(p(s)))]ds, \tag{4.13}$$

which is bounded from (3.9) by

$$\int_{-T}^{0}\|(AQ)^{\frac{1}{2}}e^{AQs}\|_{op}ds\,\eta_4(h)$$
$$+ \int_{-T}^{0}\|e^{AQs}\|_{op}C_1\{|p_h(s) - p(s)|_V + |\Phi_h(p_h(s)) - \Phi(p(s))|_V\}ds.$$

We have the following in the middle of the proof of Lemma 4.3,

$$|\Phi_h(p_h(s)) - \Phi(p(s))|_V \leqq |p_h(s) - p(s)|_V + \|\Phi - \Phi_h\|_V + (|p(0)|_H + \frac{C_0}{\lambda_N})e^{-\lambda_N s}N\eta_2(h).$$

Hence (4.13) is bounded by

$$2e^{-\frac{1}{2}\lambda_N^{-\frac{1}{2}}{}_{+1}}\eta_4(h) + \int_{-\tau}^{0} e^{\lambda_{N+1}s}C_1\{2|p_h(s) - p(s)|_V + \|\Phi - \Phi_h\|_V + (\frac{\rho_0}{\lambda_1^{\frac{1}{2}}} + \frac{C_0}{\lambda_N})e^{-\lambda_N s}N\eta_2(h)\}ds$$

and by Lemma 4.3, this is also bounded by

$$2e^{-\frac{1}{2}\lambda_N^{-\frac{1}{2}}{}_{+1}}\eta_4(h) + 2C_1 \int_{-\tau}^{0} e^{\lambda_{N+1}s}\{e^{-s\lambda^*}\eta_{10}(h) + \frac{2C_1}{\lambda_N}e^{-s\lambda_N}\|\Phi - \Phi_h\|_V\}ds$$
$$+ \frac{1}{\lambda_{N+1}}\|\Phi - \Phi_h\|_V + \frac{1}{\lambda_{N+1}-\lambda_N}(\frac{\rho_0}{\lambda_1^{\frac{1}{2}}} + \frac{C_0}{\lambda_N})N\eta_2(h)$$
$$= (\frac{4C_1^2}{\lambda_N(\lambda_{N+1}-\lambda_N)} + \frac{1}{\lambda_{N+1}})\|\Phi - \Phi_h\|_V + \eta_{12}(h)$$

where

$$\eta_{12}(h) = 2e^{-\frac{1}{2}\lambda_N^{-\frac{1}{2}}{}_{+1}}\eta_4(h) + \frac{2C_1\eta_{10}(h)}{\lambda_{N+1}-\lambda^*} + \frac{1}{\lambda_{N+1}-\lambda_N}[\frac{\rho_0}{\lambda_1^{\frac{1}{2}}} + \frac{C_0}{\lambda_N}]N\eta_2(h).$$

Without loss of generality, we may assume that

$$\frac{4C_1^2}{\lambda_N(\lambda_{N+1}-\lambda_N)} + \frac{1}{\lambda_{N+1}} \leqq \frac{1}{2}.$$

Finally we obtain

$$|\Phi_h(\pi_h^{-1}(\alpha)) - \Phi_0(\pi_h^{-1}(\alpha))|_V \leqq \frac{1}{2}\|\Phi - \Phi_h\|_V + \eta_{11}(h, T) + \eta_{12}(h) + \frac{C_0}{\lambda_{N+1,h}}e^{-\lambda_{N+1,h}T} + \frac{C_0}{\lambda_{N+1}}e^{-\lambda_{N+1}T}.$$

and

$$\|\Phi - \Phi_h\|_V \leqq 2\eta_{11}(h, T) + 2\eta_{12}(h) + 2\frac{C_0}{\lambda_{N+1,h}}e^{-\lambda_{N+1,h}T} + 2\frac{C_0}{\lambda_{N+1}}e^{-\lambda_{N+1}T}.$$

This completes the proof of C^0-convergence of $\{\Phi_h\}$.

4.3. C^1 convergence of $\{\Phi_h\}$

We use the similar argument to prove C^1 convergence. Recall that the derivative $D\Phi_h$ was constructed as a fixed point of the infinitesimal version of Lyapunov Perron intergral, that is,

$$D\Phi_h(p_0) = \int_{-\infty}^{0} e^{A_h Q_h s} Q_h D_u F_h(u_h(s))(P_h + D\Phi_h(p_h)) J_h(p_{h,0,s}) ds$$

where $u_h(s) = p_h(s) + \Phi_h(p_h(s))$ and $J_h(p_{h,0,t})$ is the solution of the linear differential equation

$$\dot{\rho}_t + A_h P_h \rho = P_h D_u F_h(p_h + \Phi_h(p_h))(P_h + D\Phi_h(p_h))\rho$$

that satisfies $J_h(p_{h,0}, 0) = P_h$.

Before we start proving, we recall that Φ_h is considered as a mapping from R^N to V_h by the composite function, i.e.,

$$\Phi_h(\alpha) = \Phi_h(\pi_h^{-1}(\alpha)).$$

Hence its derivative is to be

$$D\Phi_h(\alpha)\beta = D\Phi_h(\pi_h^{-1}(\alpha))\pi_h^{-1}(\beta)$$

and we need to estimate ;

$$\begin{aligned} (4.14) \qquad & D\Phi_h(\alpha)\beta - D\Phi(\alpha)\beta \\ &= D\Phi_h(\pi_h^{-1}(\alpha))\pi_h^{-1}(\beta) - D\Phi(\pi^{-1}(\alpha))\pi^{-1}(\beta). \end{aligned}$$

The next result is the infinitesimal version of Lemma 4.3, but much more complicated.

Lemma 4.4. *Let $J_h(\pi_h^{-1}(\alpha), t)$ be the solution of the linear differential equation*

$$(4.15) \qquad \rho_t + A_h P_h \rho = P_h D_u F_h(p_h + \Phi_h(p_h))(P_h + D\Phi_h(p_h))\rho$$

that satisfies $J_h(\pi_h^{-1}(\alpha), 0) = P_h$, then for $|\alpha|_{R^N} \leqq \rho_0$,

$$\begin{aligned} &|J_h(\pi_h^{-1}(\alpha), t)\pi_h^{-1}(\beta) - J(\pi^{-1}(\alpha), t)\pi^{-1}(\beta)|_V \\ &\leqq e^{-(\lambda^* + C_1)t}\eta_{13}(h, t) + 3e^{-(\lambda_N + C_1)t}\rho_0 \|D\Phi_h - D\Phi\|_V \end{aligned}$$

where $t < 0$ and $\eta_{13}(h, t) \to 0$ as $h \to 0$ uniformly on bounded t.

Proof. From the variation of constant formula, we have

$$J_h(\pi_h^{-1}(\alpha), t)\pi_h^{-1}(\beta) = e^{-tA_h P_h} P_h \pi_h^{-1}(\beta)$$

$$- \int_t^0 e^{(s-t)A_hP_h}P_hD_uF_h(u_h(s))(P_h + D\Phi_h(p_h))J_h(\pi_h^{-1}(\alpha), s)\pi_h^{-1}(\beta)ds$$

and by using the previous Lemmas and Gronwall inequality we can prove this lemma. We skip its proof for conciseness.

Not we turn to the estimations of (4.14).

(4.16)

$$\begin{aligned} &D\Phi_h(\pi_h^{-1}(\alpha))\pi_h^{-1}(\beta) - D\Phi(\pi^{-1}(\alpha))\pi^{-1}(\beta) \\ &= \int_{-\infty}^0 e^{A_hQ_hs}Q_hD_uF_h(u_h(s))(P_h + D\Phi_h(p_h))J_h(p_{h,0}, s)ds\pi_h^{-1}(\beta) \\ &- \int_{-\infty}^0 e^{AQs}QD_uF(u(s))(P + D\Phi(p))J_h(p_0, s)ds(\pi^{-1}(\beta)). \end{aligned}$$

We split the integral into two part $\int_{-\infty}^{-T}$ and $\int_{-T}^{0}$. We use Lemma 4.4 and argue as proving C^0-convergence to get a bound for the first integral. The second integral is relatively easy to handle. Altogether we get

$$\begin{aligned} &|D\Phi_h(\pi_h^{-1}(\alpha))\pi_h^{-1}(\beta) - D\Phi(\pi^{-1}(\alpha))\pi^{-1}(\beta)|_V \\ &\leqq \eta_{14}(h, T) + \frac{7\rho_0}{\lambda_{N+1}-\lambda_N-C_1}\|D\Phi_h - D\Phi\|_V \\ &+ (\frac{2C_1\rho_0}{\lambda_{N+1,h}-\lambda_{N,h}-C_1}e^{-\lambda_{N+1,h}T}) + (\frac{2C_1\rho_0}{\lambda_{N+1}-\lambda_N-C_1}e^{-\lambda_{N+1}T}), \end{aligned}$$

where $\eta_{14}(h, T) \to 0$ as $h \to 0$ for any fixed T. We may have assumed $0 \leqq \frac{7\rho_0}{\lambda_{N+1}-\lambda_N-C_1} < 1$ and we finally obtain

(4.17)

$$\begin{aligned} &(1 - \frac{7\rho_0}{\lambda_{N+1}-\lambda_N-C_1}\|D\Phi_h - D\Phi\|_V \\ &\leqq \eta_{16}(h, T) + \frac{2C_1\rho_0}{\lambda_{N+1,h}-\lambda_{N,h}-C_1}e^{-\lambda_{N+1,h}T} + \frac{2C_1\rho_0}{\lambda_{N+1}-\lambda_N-C_1}e^{-\lambda_{N+1}T}. \end{aligned}$$

This inequality implies C^1-convergence of $\{\Phi_h\}$.

5. Comparision with the Spectral Galerkin approximation

The other way to approximate the inertial manifold is by taking spectral projection of the equation (2.1) :

For any integer $M \geqq 1$ we consider the following Galerkin approximation

$$\frac{du_M}{dt} + Au_M = P_M F(u_M) \tag{5.1}$$

where u_M takes its value in $P_M V = P_M \mathcal{D}(A^{\frac{1}{2}})$. As it already remarked in Foias, Sell, Temam (1988), (5.1) satisfies the same properties as (2.1). Consequently, for every M large enough, one can construct an inertial manifold $\mathcal{M}_M$ which is the graph of a smooth function

$$\Phi_M : P_N V \rightarrow Q_N P_M V \subset Q_N V.$$

Moreover

$$\|\Phi - \Phi_M\|_V = \sup_{p \in P_N V} |\Phi(p) - \Phi_M(p)|_V = \mathcal{O}(\lambda_{M+1}^{-\frac{1}{2}}) .$$

Furthermore, the eigenvalues λ_n of the operator A have the asymptotic representation

$$\lambda_n = D_n + o(n^{\frac{1}{2}}), \quad \text{as} \quad n \rightarrow \infty$$

for some constant D, see Sell (1989).

Compared to these, we have in Theorem 2.1

$$\|\Phi - \Phi_h\| = \mathcal{O}(h)$$

and as mesh size h decreases the dimention, say M, of the approximate equations increases in proportion to $\frac{1}{h^2}$ that is

$$\|\Phi - \Phi_h\| = \mathcal{O}(\frac{1}{\sqrt{M}}).$$

Therefore in both cases, we get the same asymptotic convergences. In other words, we need asymptotically same number of approximate equations to get a good approximation.

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