HONAM MATHEMATICAL JOURNAL
Volume 14, Number 1. July 1992.

A Duality for Contravariant Functors on the Category Ban*

Won-Kee Jeon

*Department of Mathematics, Chonbuk National University,
Chonju, Chonbuk 560—756, Korea.*

Let Ban denote the class of all Banach spaces over the complexes $\mathbb{C}$. Then there are two important categories connected with Ban.

One category Ban_∞ consists of all Banach spaces in Ban and all bounded linear maps between Banach spaces.

The other category Ban_1 has the same objects as Ban_∞ (i.e., $\text{Obj}(\text{Ban}_1) = \text{Obj}(\text{Ban}_\infty)$) and the morphisms of Ban_1 consists only of all linear contractions (i.e., bounded linear maps φ satisfying $\|\varphi\| \leq 1$) between Banach spaces. The category Ban_1 has the advantage that in it all limits and colimits exist ([1]).

We shall use the abbreviation "Category Ban" to mean either Ban_1 or Ban_∞ if some statements hold for both categories.

Let $\underline{K}$ be a full subcategory of Ban. The purpose of this paper is to define a duality $D : \text{Ban}^{\underline{K}} \longrightarrow (\text{Ban}^{\underline{K}})^{op}$ for contravariant functors which is admissible (linear and contractive on Hom-spaces) and self-adjoint on the right (Definition 4) and to prove some properties of the dualities D defined as above (Theorem 5 and 6), where

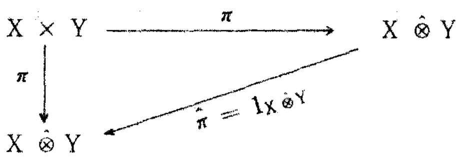
$\text{Obj}(\text{Ban}^{\underline{K}})$ = the class of all functors from $\underline{K}$ to Ban and $\text{Morph}(\text{Ban}^{\underline{K}})$ = the class of all natural transformations between functors, and $\underline{K}^{op}$ is the opposite category of the category $\underline{K}$.

For X, Y $\in$ Obj(Ban) a projective tensor product of X and Y is a Banach space $X \hat{\otimes} Y$ together with a bounded bilinear map $\pi : X \times Y \longrightarrow X \hat{\otimes} Y$ such that for any bounded bilinear map $g : X \times Y \longrightarrow Z$(an arbitrary Banach space) there exists a unique bounded linear map $\hat{\varphi} : X \hat{\otimes} Y \longrightarrow Z$ with $\varphi = \hat{\varphi} \circ \pi$ and $\|\varphi\| = \|\hat{\varphi}\|$ ([3], [4], [5]).

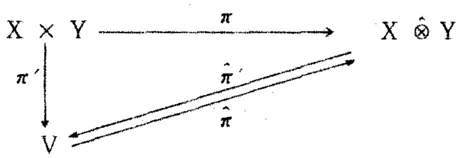
Proposition 1. If a projective tensor product exists it is uniquely determined up to isometrical isomorphisms.

* This paper was supported by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1991.

Proof. Taking $Z = X \hat{\otimes} Y$ and $\varphi = \pi$ we have the following commutative diagram



and thus $\|\pi\| = \|\hat{\pi}\| \leqq 1$. Suppose that there is a second tensor product (V, π') of X and Y. Then, in the commutative diagram



$\pi' = \hat{\pi}' \circ \pi$ and $\pi = \hat{\pi} \circ \pi'$. Since the factorization is unique, $\hat{\pi} \circ \hat{\pi}' = 1_{X \hat{\otimes} Y}$ and $\hat{\pi}' \circ \hat{\pi} = 1_V$. Thus

$$\|\pi'\| = \|\hat{\pi}'\| \leqq 1 \quad \text{and} \quad \|\pi\| = \|\hat{\pi}\| \leqq 1$$

and thus V and $X \hat{\otimes} Y$ are isometrically isomorphic. ///

Note that $X \otimes Y$ means the algebraic tensor product $X \otimes_{\mathbb{C}} Y$ over $\mathbb{C}$ and for each $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \hat{\otimes} Y$

$$\|u\|_{X \hat{\otimes} Y} = \|u\|^{\wedge} = \inf \sum_{i=1}^{n} \|x_i\| \|y_i\| .$$

Moreover, $X \otimes Y$ is a dense subspace of $X \hat{\otimes} Y$ with norm $\| \ \|^{\wedge}$ ([1]).

The set of all morphisms from X to Y in the category Ban_{∞} coincides with the Banach space H(X, Y) of all bounded linear maps from X to Y, whereas the set of morphisms Hom(X, Y) in Ban_1 consists of the unit ball $\{ f \in H(X, Y) \mid \|f\| \leqq 1 \} \subset H(X, Y)$.

The above Hom-functor H is a contra-covariant bifunctor into Ban and we will often consider its covariant partial functor $H_A = H(A, -)$, the action on morphisms being given by

$$H(A, f)g = f \circ g \quad \text{for} \quad f: X \longrightarrow Y \quad \text{and} \quad g \in H(A, X),$$

and the contravariant partial functors $H^A = H(-, A)$, the action on morphisms being given by

$$H(f, A)g = g \circ f \quad \text{for} \quad f: X \longrightarrow Y \quad \text{and} \quad g \in H(Y, A).$$

For the co-covariant bifunctor $\hat{\otimes}$ defined by the projective tensor product $X \hat{\otimes} Y$ in Ban, its partial functor $X \hat{\otimes} -$ acts on morphisms by

$$(X \hat{\otimes} f)(\sum x_i \otimes y_i) = \sum x_i \otimes f(y_i) \quad \text{for} \quad f : Y \longrightarrow Z.$$

For Banach spaces X, Y and Z we have an isometrically isomorphism

$$H(X \hat{\otimes} Y, Z) \simeq H(X, H(Y, Z))$$

where $\hat{\varphi} \longleftrightarrow \varphi$ with $\varphi(x)(y) = \hat{\varphi}(x \otimes y)$ for $x \in X$ and $y \in Y$.

Therefore the contravariant functor H^Z is adjoint on the right to itself, that is,

$$\begin{aligned} H(X, H(Y, Z)) &= H(X \hat{\otimes} Y, Z) = H(Y \hat{\otimes} X, Z) \\ &= H(Y, H(X, Z)). \end{aligned}$$

In particular, H^Z transforms colimits into limits in Ban_1, a special case being $(\varinjlim X_d)' = H(\varinjlim X_d, \mathbf{C}) = \varprojlim (X_d, \mathbf{C})$.

Proposition 2. If a functor $F : \mathrm{Ban} \longrightarrow \mathrm{Ban}$ commutes with colimts, then $F(-) = (-) \hat{\otimes} F(\mathbf{C})$. On the other hand, if a contravariant functor $G : \mathrm{Ban}^{op} \longrightarrow \mathrm{Ban}$ transforms colimits into limits, then $G(-) = H(-, G(\mathbf{C}))$.

Proof. Let $\{X_s \mid s \in S\}$ be a family of Banach spaces, where S is an arbitrary index set. If the product of this family exists, then it is also a Banach space ([1]) and denoted by $\prod_{s \in S} X_s$. If $X_s = X$ for all $s \in S$, then we put $\prod_{s \in S} X_s = l_S^{\infty}(X)$. In particular, if $X = \mathbf{C}$ then we put such that

$$l_S^{\infty}(X) = l_S^{\infty}(\mathbf{C}) = l_S^{\infty}.$$

We use the notation

$$l_S(X) = l_S^1(\mathbf{C}) = l_S^1$$

for the coproduct of a family $\{X_S \mid s \in S\}$ of Banach spaces.

We have to note that every Banach space X may be represented as a colimit of space l_n^1, i.e., $X = \varinjlim l_n^1$ which is naturally in X, where n is a positive integer ([1], [3]). Then we have the following :

$$\begin{aligned} F(x) &= F(\varinjlim l_n^1) = \varinjlim (F(l_n^1)) \\ &= \varinjlim F(\mathbf{C} \oplus \cdots \oplus \mathbf{C}) \ (n-\text{times}) \\ &= \varinjlim (F(\mathbf{C}) \oplus \cdots \oplus F(\mathbf{C})) \ (n-\text{times}) \\ &= \varinjlim (\mathbf{C} \hat{\otimes} F(\mathbf{C}) \oplus \cdots \oplus \mathbf{C} \hat{\otimes} F(\mathbf{C})) \ (n-\text{times}) \end{aligned}$$

$$= \varinjlim (l_n^1 \hat{\otimes} F(\mathbb{C}))$$
$$= \varinjlim l_n^1 \hat{\otimes} F(\mathbb{C}) \ (X \hat{\otimes} - \text{ commutes with colimits})$$
$$= X \hat{\otimes} F(\mathbb{C}).$$

$$G(X) = G(\varinjlim l_n^1) = \varprojlim G(l_n^1) = \varprojlim l_n^\infty (G(\mathbb{C}))$$
$$= \varprojlim H(l_n^1, G(\mathbb{C}))$$
$$= H(\varinjlim l_n^1, G(\mathbb{C})) \ (\text{by the property of } H)$$
$$= H(X, G(\mathbb{C})). \qquad ///$$

For two functors F and F_1 from Ban to Ban a natural transformation $\alpha : F \longrightarrow F_1$ is a family of morphism $\alpha_X : F(X) \longrightarrow F_1(X) \in$ Morph(Ban) satisfying the commutative diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\alpha_X} & F_1(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle F_1(f)} \\ F(Y) & \xrightarrow{\alpha_Y} & F_1(Y) \end{array}$$

for a morphism $f : X \longrightarrow Y$ in Morph(Ban) and farthermore

$$\|\alpha\| = \sup_{X \in Obj(Ban)} \|\alpha_X\| < \infty.$$

By Nat(F, F_1) we means the Banach space of all natural transformations $F \longrightarrow F_1$ with coordinate-wise operators.

The unit ball of Nat(F, F_1) is the set of all natural transformations $F \longrightarrow F_1$ for functors F, F_1 : $Ban_1 \longrightarrow Ban_1$.

Lemma 3. For functors F : Ban $\longrightarrow$ Ban and contravariant functor G : $Ban^{op} \longrightarrow$ Ban we have

$$Nat(H_A, F) = F(A) \quad \text{and} \quad Nat(H^A, G) = G(A),$$

for all Banach space A.

Proof. For each $\varphi \in Nat(H_A, F)$ we have a morphism $\varphi_A : H(A, A) \longrightarrow F(A)$. Then for $1_A \in H(A, A)$, $\varphi_A(1_A) = f_A \in F(A)$.

Thus we define

$$\eta : Nat(H_A, F) \longrightarrow F(A)$$

by $\eta(\varphi) = \varphi_A(1_A) = f_A$.

On the other hand, η^{-1} is defined by

$$\begin{array}{cccc} \eta^{-1}(f_A)_X : & H(A, X) & \longrightarrow & F(X) . \\ & \Cup & & \Cup \\ & f & \longrightarrow & \eta^{-1}(F_A)_X(f) = F(f)f_A . \end{array}$$

Then by the commutative diagram

$$\begin{array}{ccc} H(A, A) & \xrightarrow{\varphi_A} & F(A) \\ {\scriptstyle H(A, f)} \downarrow & & \downarrow {\scriptstyle F(f)} \\ H(A, X) & \xrightarrow{\varphi_X} & F(X) \end{array}$$

it is clear that $\eta^{-1}(f_A)_X(f) = \varphi_X(f) = F(f)f_A$.

Similarly, for each $\psi \in Nat(H^A, G)$

$$\begin{array}{cccc} \xi : & Nat(H^A, G) & \longrightarrow & G(A) \\ & \Cup & & \Cup \\ & \psi & \longmapsto & \psi_A(1_A) \end{array}$$

and for each $g_A \in G(A)$ $\xi^{-1}(g_A)_X(g) = G(g)(g_A)$ for $g : A \longrightarrow X$ in Morph(Ban).

/ / /

Definition 4. A duality for contravariant functors in a covariant functor

$$D : Ban^{\underline{K}} \longrightarrow (Ban^{\underline{K}})^{op}$$

which is linear and contractive on Hom-spaces and self-adjoint on the right, i.e.,

$$\eta_{F_1F_2} : \underset{\underline{K}}{Nat} (F_1, DF_2) \simeq \underset{\underline{K}}{Nat} (F_2, DF_1)$$

holds naturally in F_1 and F_2 via an isometric isomorphism $\eta_{F_1F_2}$ with $\eta_{F_2F_1} = \eta_{F_1F_2}^{-1}$, where $\underline{K}$ is a full subcategory of Ban.

Let $G : \underline{K}^{op} \times \underline{K}^{op} \longrightarrow Ban$ be a contra-contvariant bifunctor.

G is said to be <u>symmetric</u> if there is an isometric isomorphism $t : G(X, Y) \longrightarrow G(Y, X)$ which is natural in X and Y such that $tt = 1_G$, i.e., t is an involution.

For example, $H(-, Z_1) \hat{\otimes} H(-, Z_2) : \underline{K}^{op} \times \underline{K}^{op} \longrightarrow Ban$ is a symmetric bifunctor.

Theorem 5. Let $G : \underline{K}^{op} \times \underline{K}^{op} \longrightarrow$ Ban be a symmetric bifunctor. Then $D_G : \mathrm{Ban}^{\underline{K}} \longrightarrow (\mathrm{Ban}^{\underline{K}})^{op}$, which is defined by $D_G F(X) = \underset{\underline{K}}{\mathrm{Nat}}\,(F, G(-, X))$ is a quality for contravariant functor.

Proof. We have to prove that D_G is admissible and self-adjoint on the right. For a natural transformation $f : F \longrightarrow F_1$ since

$$\mathrm{Nat}(f, G(-, X)) : \mathrm{Nat}(F, G(-, X)) \longrightarrow \mathrm{Nat}(F, G(-, X))$$

on Ban, we have

$$D_G(f)(X) : \mathrm{Nat}(F, G(-, X)) \longrightarrow \mathrm{Nat}(F_1, G(-, X))$$

on $(\mathrm{Ban}^{\underline{K}})^{op}$. Therefore D_G is a covariant functor on $\mathrm{Ban}^{\underline{K}}$.

Since Nat is the Hom-functor of $\mathrm{Ban}^{\underline{K}}$, D_G is obviously admissible. Thus we have to prove that D_G is adjoint to itself on the right.

$$\begin{aligned}
\mathrm{Nat}(F_1, D_G F_2) &= \underset{X}{\mathrm{Nat}}\,(F_1(X), D_G F_2(X)) \\
&= \underset{X}{\mathrm{Nat}}\,(F_1(X), \underset{Y}{\mathrm{Nat}}\,(F_2(Y), G(Y, X))) \\
&= \underset{X}{\mathrm{Nat}}\,(F_1(X) \,\hat{\otimes}_Y\, F_2(Y), G(Y, X)) \\
&= \underset{X}{\mathrm{Nat}}\,(F_2(Y) \,\hat{\otimes}\, F_1(X), G(X, Y)) \\
&= \underset{Y}{\mathrm{Nat}}\,(F_2(Y), \underset{X}{\mathrm{Nat}}\,(F_1(X), G(X, Y))) \\
&= \underset{Y}{\mathrm{Nat}}\,(F_2(Y), D_G F_1(Y)) \\
&= \mathrm{Nat}\,(F_2, D_G F_1)\ . \qquad ///
\end{aligned}$$

Theorem 6. We have a one-to-one correspondence between dualities D on $\mathrm{Ban}^{\underline{K}}$ for contravariant functors and contra-contravariant symmetric functors $G : \underline{K}^{op} \times \underline{K}^{op} \longrightarrow$ Ban.

Thus our duality D has the form D_G of Theorem 5.

Proof. Let D be a duality for contravariant functors. For $X, Y \in \mathrm{Obj(Ban)}$ we define

$$G^D(X, Y) = DH^Y(X).$$

It is clearly a contravariant functor in X and since $g : Y_1 \longrightarrow Y_2$ define a natural transformation

$$H^2 : H^{Y_1} \longrightarrow H^{Y_2}\ .$$

on $\mathrm{Ban}^{\underline{K}}$, we have

$$G^D(X, g) : DH^{Y_2}(X) \longrightarrow DH^{Y_1}(X)$$

on $(\mathrm{Ban}^{\underline{K}})^{op}$. Thus, $G^D(X, -)$ is a contravariant functor. Moreover, since $DH^Y \in \mathrm{Obj}(\mathrm{Ban}^{\underline{K}})^{op}$, for each morphism $f : X_1 \longrightarrow X_2 \in \mathrm{Morph}(\mathrm{Ban})$

$$DH^Y(f) : DH^Y(X_2) \longrightarrow DH^X(X_1),$$

and thus $G^D(f, Y) : G^D(X_2, Y) \longrightarrow G^D(X_1, Y)$. That is, G^D is a contra-contravariant bifunctor satisfying the commutative diagram :

$$\begin{array}{ccc} G^D(X_2, Y_2) & \xrightarrow{\quad G^D(X_2,\, g) \quad} & G^D(X_2, Y_1) \\ {\scriptstyle G^D(f,\, Y_2)} \downarrow & © & \downarrow {\scriptstyle G^D(f,\, Y_1)} \\ G^D(X_1, Y_2) & \xrightarrow{\quad G^D(X_1,\, g) \quad} & G^D(X_2, Y_2) \end{array}$$

for $f : X_1 \longrightarrow X_2$ and $g : Y_1 \longrightarrow Y_2$.

By Lemma 3 we have

$$\begin{aligned} G^D(X, Y) &= \mathrm{Nat}(H^X, DH^Y) \\ &= \mathrm{Nat}(H^Y, DH^X) \quad \text{(D is self-adjoint)} \\ &= DH^X(Y) \quad \text{(by Lemma 3)} \\ &= G^D(Y, X) \end{aligned}$$

Hence G^D is symmetric and natural in X and Y.

On the other hand, we have $G^D = G$ for each symmetric bifunctor G on $\underline{K}^{op} \times \underline{K}^{op}$, because of that

$$\begin{aligned} G^{D_G}(X, Y) &= D_G H^Y(X) \quad \text{(by the above definition)} \\ &= \mathrm{Nat}(H^Y, G(-, X)) \quad \text{(by Theorem 5)} \\ &= G(Y, X) \quad \text{(by Lemma)} \\ &= G(X, Y) \quad \text{(G is symmetric)} \end{aligned}$$

Moreover, $D_{G^D} = D$ because of that

$$\begin{aligned} D_{G^D}F(X) &= \mathrm{Nat}(F, G^D(-, X)) \quad \text{(by Theorem 5)} \\ &= \mathrm{Nat}(F, DH^X) \quad \text{(by the above definition)} \\ &= \mathrm{Nat}(H^X, DF) \quad \text{(D is self-adjoint)} \\ &= DF(X) \quad \text{(by Lemma 3)} \end{aligned}$$

Therefore D $\longleftrightarrow$ D_G $\longleftrightarrow$ G and thus we complete our proof. ///

References

[1] J. Cigler, V. Losert and P. Michor : Banach Modules and Functors on Categories of Banach Spaces, Marcel Dekker, INC. (1979).
[2] E. J. Dubuc : Kan Extensions in Enriched Category Theory, Springer Lecture Notes 145 (1970).
[3] C. Herz and J. W. Pelletier : Dual Functors and Integral Operators in the Category of Banach Spaces, Preprint, Vienna, (1973).
[4] J. W. Negrepontis : Duality of functions on Categories of Banach spaces, *J. Pure Appl. Algebra* 3 (1973) pp. 119~131.
[5] K. L. Pothoven : Compact functors and their duals in Categories of Banach Spaces, *Trans. AMS 155* (1971) pp. 148~159.