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On Characterizations of a Real Hypersurface of Type A in a Complex Space Form*

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Introduction

A complex n-dimensional Kaehler manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $M_n(c)$. A complete and simply connected complex space form consists of a complex projective space P_nC, a complex Euclidean space C^m or a complex hyperbolic space H_nC, according as $c > 0$, $c = 0$ or $c < 0$.

Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the Kaehler metric and the almost complex structure J of $M_n(c)$. We denote by ∇, A, S, and S' the Riemannian connection, the shape operator, the Ricci tensor of type (1, 1) and the of type (0, 2) of M, respectively. The second fundamental form H of M is defined by $g(AX, Y) = H(X, Y)$ for any vector fields X and Y. Many differential geometers have studied real hypersurfaces of a complex projective space (cf. [2], [7], [8], [9], [16] and [17] by using the structure (ϕ, ξ, η, g). Typical examples of a real hypersurface in P_nC are homogeneous ones. Takagi [16] showed that all homogeneous real hypersurfaces in P_nC are realized as the tubes of constant radius over Kaehler submanifolds. Namely, he proved the following

Theorem A ([16]). Let M be a homogeneous real hypersurface of P_nC. Then M is a tube of radius γ over one of the following Kaehler submanifolds:

(A_1) a hyperplane $P_{n-1}C$, where $0 < \gamma < \frac{\pi}{2}$,

(A_2) a totally geodesic P_kC $(1 \leq k \leq n-2)$, where $0 < \gamma < \frac{\pi}{2}$,

(B) a complex guadric Q_{n-1}, where $0 < \gamma < \frac{\pi}{4}$,

(C) $P_1C \times P_{(n-1)/2}C$, where $0 < \gamma < \frac{\pi}{4}$ and $n (\geq 5)$ is odd,

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(D) a complex Grassmann $G_{2,5}C$, where $0 < \gamma < \frac{\pi}{4}$ and $n = 9$,

(E) a Hermitian symmetric space SO(10) / U(5), where $0 < \gamma < \frac{\pi}{4}$ and $n = 15$.

On the other hand, real hypersurfaces of a complex hyperbolic space H_nC have also investigated by Berndt [1], Montiel [13] and Montiel and Romero [14] and so on. Berndt [1], Montiel [13] and Montiel and Romero [14] and so on. Berndt [1] classified all homogeneous real hypersurfaces of H_nC and showed that they are realized as the tubes of constant radius over Kaehler submanifolds. Namely, he proved the following

Theorem B ([1]). Let M be a homogeneous real hypersurface of H_nC. Then M is a tube of radious γ over one of the following Kaehler submanifold :

(A_0) a horosphere in H_nC,

(A_1) a complex hyperbolic hyperplane $H_{n-1}C$ of arbitrary radius,

(A_2) a totally geodesic H_kC $(1 \leqq k \leqq n-2)$ of arbitrary radius,

(B) a totally real hyperbolic space RH_n.

It is proved in [14] and [15] (see also [3]) that in order for a real hypersurface of $M_n(c)$, $c \neq 0$ to be of type A_1, A_2 if $c > 0$ or of type A_0, A_1, A_2 if $c < 0$, it is necessary and sufficient that it satisfies $L_\xi g = 0$, where L_ξ denotes the Lie derivative in the direction of the structure vector field ξ. For simplicity, we shall say that a real hypersurface M of $M_n(c)$, $c \neq 0$ is of *type A* if it is of type A_1 or type A_2 in P_nC, or it is of type A_0, of type A_1 or type A_2 in H_nC. Giving attention to this fact Ki, Kim and Lee [4], Lee [10] and Maeda and Udagawa [11] proved the following

Theorem C. Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. Then the following are equivalent:

(1) M is of type A,

(2) $L_\xi A = 0$ ([4]),

(3) $L_\xi \nabla = 0$ ([10]),

(4) $L_\xi \phi = 0$ (11]).

The main purpose of the present paper is to give another characterizations of type A in $M_n(c)$, $c \neq 0$. Namely, we will prove

Theorem 1. Let M be a real hypersurface with real dimension $2n-1$ of $M_n(c)$, $c \neq 0$ and

$n \geqq 3$. Then M is of type A if and only if $L_\xi S = 0$ and $L_\xi H = 0$.

Theorem 2. Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. Then M is of type A if and only if $L_\xi S' = 0$ *and* $L_\xi H = 0$.

Now we prepare without proof the following in order to prove to our results :

Theorem D ([5]). Let M be a real hypersurface of $M_n(c)$, $c \neq 0$ and $n \geqq 3$. If ξ is principal and it satisfies $L_\xi S = 0$, then the mean curvature of M is constant.

1. Preliminaries

Let M be a real hypersurface of a complex n - dimensional complex space form $M_n(c)$ of constant holomorphic sectional curvature c, and let C be a unit normal vector field on a neighborhood of a point x in M, the images of X and C under the linear transformation J can be represented as

$$JX = \phi X + \eta(X)\xi, \; JC = -\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, while η and ξ denotes a 1-form and a vector field on the neighborhood of x in M, respectively. Then it is seen that $g(\xi, X) = \eta(X)$, where g denotes the induced Riemannian metric on M. The set of tensors (ϕ, ξ, η, g) is called an almost contact metric structure on M. They satisfy the following

$$\phi^2 = -I + \eta \otimes \xi, \; \phi\xi = 0, \; \eta(\phi X) = 0, \; \eta(\xi) = 1$$

for any vector field X, where I denotes the identity transformation. Furthermore, the covariant derivatives of the structure tensors are given by

$$\nabla_X \xi = \phi AX, \tag{1.1}$$

$$\nabla_X \phi(Y) = \eta(Y)AX - g(AX, Y)\xi \tag{1.2}$$

for any vector fields X and Y on M, where $\nabla$is the Riemannian connection of g and A denotes the shape operator in the direction of C on M. The second fundamental form H of M is given by $g(AM, Y) = H(X, Y)$ for any vector fields X and Y.

Since the ambient space is of constant holomorphic sectional curvature c, the equations of the Gauss and Codazzi are respectively obtained [4] :

$$(1.3)\qquad R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\}/4 + g(AY, Z)AX - g(AX, Z)AY,$$

$$(1.4)\qquad \nabla_X A(Y) - \nabla_Y A(X) = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}/4,$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X.

The Ricci tensor S' of M is a tensor field of type (0, 2) given by $S'(X, Y) = tr\{Z \rightarrow R(Z, X)Y\}$. Also it may be regarded as the tensor field of type (1, 1) and denoted by $S : TM \rightarrow TM$; it satisfies $S'(X, Y) = g(SX, Y)$. From (1. 3) we see that the Ricci tensor S of M is given by

$$(1.5)\qquad S = c\{(2n+1)I - 3\eta\otimes\xi\}/4 + hA - A^2$$

where we have put $h = trA$. By (1.1) and (1.5), we find

$$(1.6)\qquad \nabla_X S(Y) = -3c\{g(\phi AX, Y)\xi + \eta(Y)\phi AX\}/4 + dh(X)AY + (hI - A)\nabla_X A(Y) - \nabla_X A(AY),$$

where d denotes the exterior differential.

Suppose that the structure vector field ξ is principal with corresponding principal curvature α. Then it is well known that α is constant on M ([6] and [12]), and it satisfies

$$(1.7)\qquad 2A\phi A = \frac{c}{2}\phi + \alpha(A\phi + \phi A).$$

Thus, if X is a principal vector with corresponding principal curvature λ, then we have

$$(1.8)\qquad (2\lambda - \alpha)A\phi X = (\frac{c}{2} + \alpha\lambda)\phi X.$$

From the equation (1. 1) we get

$$(\nabla_X A)\xi = \alpha\phi AX - A\phi AX,$$

with which together (1.4) and (1.7) implies that

$$\nabla_{\xi}A(X) = -\frac{\alpha}{2}(A\phi - \phi A)X. \tag{1.9}$$

Therefore, it follows that we obtain

$$dh(\xi) = 0.$$

2. Proof of Theorem 1

Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. By definition the Lie derivative of the second fundamental form with respect to ξ is given by

$$(L_{\xi}H)\ (X,\ Y) = L_{\xi}H(X,\ Y) - H(L_{\xi}X,\ Y) - H(X,\ L_{\xi}Y)$$

for any vector fields X and Y, with which together (1.1) implies that

$$L_{\xi}H = \nabla_{\xi}H. \tag{2.1}$$

The Lie derivative of the Ricci tensor S of type (1,1) is also obtained:

$$L_{\xi}S(X) = L_{\xi}(SX) - SL_{\xi}X$$

for any vector field X, or using (1.1) and (1.6)

$$\begin{aligned} L_{\xi}S(X) = & -3c\{g(\phi A\xi,\ X)\xi + \eta(X)\phi A\xi\}/4 + dh(\xi)AX \\ & + (hI - A)\nabla_{\xi}A(X) - \nabla_{\xi}A(AX) - (\phi AS - S\phi A)X. \end{aligned} \tag{2.2}$$

Suppose that M is of type A, then we have $\phi A = A\phi$ and hence ξ is principal. Thus, the right hand side of (2.1) vanishies identically because of (1.9). Making use of (1.5), (1.7), (1.9) and (1.10), it is easily checked that the right hand one of (2.2) is also vanishies. Accordingly, the only part of Theorem 1 is established.

We are now going to prove the if part of Theorem 1. Suppose that $L_{\xi}S = 0$ and $\nabla_{\xi}H = 0$.

We then see from (2.1) that $\nabla_\xi A = 0$. This implies that ξ is principal (See, Lemma 4 of [4]). Consequently, by the hypothesis $L_\xi S = 0$ we obtain

(2.3) $$\frac{\alpha}{2}\{h(A\phi - \phi A)+\phi A^2 - A^2\phi\}+\phi A(hA-A^2)-(hA-A^2)\phi A = 0,$$

where we have used (1.5), (1.6), (1.9) and (1.10). Futhermore, because of the Codazzi equation (1.4) and the fact that $\nabla_\xi A = 0$, we have

$$(\nabla_X A)\xi = -\frac{c}{4}\phi X$$

for any vector field X, or taking account of (1.9)

(2.4) $$\alpha(A\phi - \phi A) = 0.$$

Therefore, (2.2) turns out to be

$$\phi A(hA-A^2)-(hA-A^2)\phi A = 0,$$

with which together (1.7) and (2.4) gives

(2.5) $$\phi A^3 - (\frac{\alpha}{2}+h)\phi A^2 - (\frac{1}{2}\alpha^2 - \alpha h)\phi A - \frac{c}{4}A\phi - \frac{c}{4}(\frac{\alpha}{2}-h)\phi = 0.$$

Since α is constant, we may only, using (2.4), consider that $\alpha = 0$ because if $A\phi = \phi A$ holds of M, then M is of type A. In this case (2.5) is reduced to

$$\phi A^3 - h\phi A^2 - \frac{c}{4}A\phi + \frac{c}{4}h\phi = 0.$$

By the properties of the almost contact metric structure and (1.7), it leads to

(2.6) $$A^4 - hA^3 + \frac{c}{4}hA - (\frac{c}{4})^2 (I - \eta\otimes\xi) = 0.$$

Let $AX = \lambda X$ for any vector field X orthogonal to ξ. By (2.6) we then have

$$(\lambda^2 - \frac{c}{4})(\lambda^2 - h\lambda + \frac{c}{4}) = 0.$$

Since the mean curvature of M is constant because of Theorem D, we see that all principal curvatures of M are constant. Therefore, by the classification theorems due to Berndt [1] and Takagi [16], M is locally congruent to one of the homogeneous real hypersurfaces of $M_n(c)$, $c \neq 0$. However, the homogeneous ones of type B, C, D and E cannot occur because of $\alpha = 0$. We remark that a homogeneous real hypersurface M with $A\xi = 0$ lies on a tube a radius $\pi/4$ over a totally geodesic P_kC, $1 \leqq k \leqq n-1$ (cf. [17]). This completes the proof of Theorem 1.

3. Proof of Theorem 2

Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. By definition the Lie derivative of the Ricci tensor S' of type (0, 2) with respect to the structure vector field ξ is given by

$$(L_\xi S')(X, Y) = L_\xi S'(X, Y) - S'(L_\xi X, Y) - S'(X, L_\xi Y)$$

for any vector fields X and Y. Because of (1.1), it turns out to be

$$(L_\xi S')(X, Y) = g(\nabla_\xi S(X) - (\phi AS + S\phi A)X, Y) \tag{3.1}$$

Suppose that $\phi A = A\phi$ holds on M. Then we see that ξ is principal. So we have $\nabla_\xi S = 0$, where we have used (1.6), (1.9) and (1.10). Therefore (3.1) is reduced to

$$(L_\xi S')(X, Y) = -g(A^2\phi AX, Y) + g(A^2\phi AX, Y) + g(A\phi A^2X, Y)$$

because of (1.5), which means $L_\xi S' = 0$. Moreover, from Theorem 1 $L_\xi H = 0$.

Conversely, if we suppose that $L_\xi S' = 0$ and $\nabla_\xi H = 0$ hold on M, then we have (2.4) and

$$\begin{aligned} &-3c\{g(\phi A\xi, X)\xi + \eta(X)\phi A\xi\}/4 + dh(\xi)AX \\ &+ (hI - A)\nabla_\xi A(X) - \nabla_\xi A(AX) - (\phi AS + S\phi A)X = 0 \end{aligned} \tag{3.2}$$

because of (1.6). In the proof of Theorem 1, we see that ξ is principal because $\nabla_\xi H = 0$ is assumed. Thus, making use of (1.5), (1.7), (1.9) and (1.10), the last equation (3.2) is reduced to $\{h\alpha + (n+1)c\}(A\phi - \phi A) = 0$. Combining this with (2.4), we obtain $A\phi = \phi A$; Hence M is of type A.

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