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CONTINUOUS GROUPS OVER THEIR ENDOMORPHISM RINGS

JAE MYUNG CHUNG

Let R be a ring and M be a left R-module. If M is (quasi-) injective, then it satisfies the following conditions [6]:

(C_1) Every submodule of M is essential in a direct summand of M;
(C_2) If a submodule A of M is isomorphic to a direct summand of M, then A is a direct summand of M.

Also if M has (C_2), then it satisfies the following condition;

(C_3) If M_1 and M_2 are direct summands of M such that $M_1 \cap M_2 = (0)$, then $M_1 \oplus M_2$ is a direct summand of M.

Mohamed and Bouhy [7] defined a module M to be continuous if it has (C_1) and (C_2); M to be quasi-continuous [8] if it has (C_1) and (C_3). Fieldhouse [3] and Ware [9] extended the notion of von Neumann regularity to modules and Zelmanowitz [10] developed following equivalent conditions of regularity:

THEOREM. [10. Theorem 2.2]. *For an R-module M, the following conditions are equivalent:*

(1) *M is regular.*
(2) *For every $m \in M$, Rm is projective and a direct summand of M.*
(3) *For every $m_1, \cdots, m_t \in M$, $\sum_{i=1}^{t} Rm_i$ is projective and a direct summand of M.*

In [4], injectivity, quasi-injectivity and continuity are the same concepts for the torsion free abelian groups (as Z-modules). But the above are distinguished for the groups as modules over their endomorphism rings. And it is easy to show that every abelian group is not regular over Z but some groups are regular over their endomorphism rings.

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In this paper we discuss injectivity, projectivity, quasi-injectivity, continuity and regularity of groups over their endomorphism rings.

Throughout this paper let G be a torsion free abelian group of finite rank n, E denote the endomorphism ring of G and S donote the center of E. We note that S is the endomorphism ring of E-module G.

A submodule N of M is closed if it has no proper essential extensions in M. Consider the following condition:

(C_1') Every closed submodule of M is a direct summand of M.

Then a module M has (C_1) if and only if it has (C_1') [6, Proposition 2.4] and we have the following result.

PROPOSITION 1. *Continuity and quasi-continuity are inherited by closed submodules.*

We call a module (ring) finite dimensional if it contains no infinite direct sums of submodules (left ideals). An E-module G is finite dimensional since G is a group of rank n.

In general, a direct sum of regular modules is regular [10, Theorem 2.8] but a direct sum of continuous modules need not be continuous [7, Example 2.5]. Therefore a regular module need not be continuous. However, we have the following results for an E-module G.

PROPOSITION 2. *Let G be regular over E. Then every submodule of $_EG$ is finitely generated.*

Proof. Let X be a submodule of G. Then X is a fully invariant subgroup of G and rank $X = k \leq n$. Let $\{x_1, x_2, \ldots, x_k\}$ be a maximal Z-independent subset of X and $A = Ex_1 + Ex_2 + \cdots + Ex_k$. Then A is a finitely generated projective submodule of X and $A \oplus B = G$ for some submodule B of G by [10, Theorem 2.2]. For any $x \in X$, $x = a + b$ for some $a \in A$ and $b \in B$. Since $\{x_1, x_2, \ldots, x_k\}$ is a maximal Z-independent subset of X, $mx \in A$ for some $0 \neq m \in Z$. From $mx - ma = mb \in A \cap B = (0)$, we have $b = 0$ and $x = a \in A$. Hence $X = A$ is finitely generated over E.

COROLLARY 2.1. *Let G be regular over E. Then*

(1) *Every submodule of $_EG$ is projective and a direct summand of G.*

(2) *G is quasi-injective (hence continuous) over E.*

Proof. (1) Obvious.

(2) Since every submodule X of G is a direct summand of G, any E-homomorphism $f : X \to G$ can be extended to a homomorphism $\overline{f} : G \to G$. Thus G is quasi-injective over E.

COROLLARY 2.2. *Let G be regular over E. Then S is a regular ring.*

Proof. Apply [10, Theorem 3.4] and S = center of E.

PROPOSITION 3. *Let G be regular over E. If A and B are submodules of G such that $A \cong B$, then $A = B$.*

Proof. By Corollary 2.1.(1), A and B are direct summands of G. Thus $A = eG$ and $B = fG$ for some idempotents e and f in S. Since $\mathrm{Hom}(eG, fG)$ and fSe are isomorphic, there exist $\alpha \in fSe$ and $\beta \in eSf$ such that $\beta\alpha = e$ and $\alpha\beta = f$. Then $e = \beta\alpha = \beta f \alpha = f\beta\alpha = fe = \alpha\beta e = \alpha e \beta = \alpha\beta = f$. Therefore $A = B$.

PROPOSITION 4. *Let G be continuous over E. Then every monomorphism in S is an automorphism.*

Proof. For any maximal Z-independent subset $\{x_1, x_2, \ldots, x_n\}$ of G, $Ex_1 + Ex_2 + \cdots + Ex_n$ is essential in G. Let $f \in S$ be a monomorphism. Then $\{fx_1, fx_2, \ldots, fx_n\}$ is a Z-independent subset of G and so $Efx_1 + Efx_2 + \cdots + Efx_n$ is essential in G. Hence $f(Ex_1 + Ex_2 + \cdots + Ex_n)$ is essential in G and is contained in $f(G) = \mathrm{Im}\, f$. Therefore $\mathrm{Im}\, f$ is essential in G and f is an automorphism by [6, Lemma 3.14].

Now we consider the case that S is regular.

PROPOSITION 5. *Let S be a regular ring. Then G has (C_2) as an E-module.*

Proof. Let a submodule A of G be isomorphic to a direct summand B of G. Then $B = eG$ for some idempotent $e \in S$. For the isomorphism $f : B \to A$ and the inclusion $i : A \to G$, $\overline{f} = ife \in E$. Since $e \in S$ and $f : B \to A$ is an E-homomorphism, $h\overline{f}(x) = h(ife(x)) = hf(ex) = f(he(x)) = f(eh(x)) = (ife)h(x) = \overline{f}h(x)$ for every $h \in E$ and every $x \in G$. Hence $h\overline{f} = \overline{f}h$ for every $h \in E$, and $\overline{f} \in S$ implies that $A = \mathrm{Im}\,\overline{f}$ is a direct summand of G. Therefore G has (C_2).

COROLLAYR 5.1. *Let S be a regular ring. If G has (C_1), then G is continuous over E.*

COROLLARY 5.2. *Let S be a regular ring. If G is uniform over E, then G is continuous over E.*

PROPOSITION 6. *Let S be a regular ring and $f \in S$, If $\operatorname{Ker} f$ is essential in G, then $f = 0$.*

Proof. Since S is regular and $f \in S$, $\operatorname{Ker} f$ is a direct summand of G. But $\operatorname{Ker} f$ is essential in G. Therefore $\operatorname{Ker} f = G$ i.e. $f = 0$.

PROPOSITION 7. *Let E be a regular ring. If G is projective over E, then G is regular over E.*

Proof. For any $x \in G$, consider the following exact sequence of E-modules:

$$0 \longrightarrow Ex \xrightarrow{i} G \xrightarrow{\pi} G/Ex \longrightarrow 0$$

where i is an inclusion and π is a natural projection. Since E is regular, G/Ex is flat over E and G is projective over E by assumption. Therefore we have a homomorphism $\alpha : G \to Ex$ such that $\alpha(x) = x$ by [9, Lemma 2.2]. This means that $\alpha i = 1$ on Ex and hence Ex is a direct summand of G. Thus G is regular over E.

COROLLARY 7.1. *Let E be a regular ring. Then G is regular over E if and only if G is projective over E.*

LEMMA 8. *If G is injective over E, then $E \cong Q_{n\times n}$ the ring of $n \times n$ matrices over Q.*

Proof. If G is injective over E, then $G = \bigoplus^n Qx_i$ for a maximal Z-independent subset $\{x_1, x_2, \ldots, x_n\}$ and $E \cong Q_{n\times n}$.

COROLLARY 8.1. *Let E be a regular ring. If G is injective over E, then G is regular over E.*

Proof. Apply Lemma 8 and Corollary 7.1.

PROPOSITION 9. *Let E be a commutative regular ring. If G is continuous over E, then $G = Qx$ for some $x \in G$.*

Proof. Since G is finite dimensional and continuous over E, $G = \bigoplus_{i=1}^k G_i$ for some indecomposable continuous submodules G_i of G [5, Proposition 1.1.9]. For each i, $G_i = e_i G$ for some primitive idempotent $e_i \in S = E$ and hence $e_i E \cong \mathrm{End}({}_E G_i)$ and $E = \bigoplus_{i=1}^{k} e_i E$. Since G_i is indecomposable and continuous, $e_i E$ is a local ring. For a non-unit $f \in e_i E$, $\mathrm{Ker}\, f$ is essential in G_i and so $f = 0$ by Proposition 6. Thus $e_i E$ is a field and E is semisimple. Therefore G is injective over E and $E \cong Q_{n\times n}$ by Lemma 8. Since E is commutative by assumption, $n = 1$ and $G = QG = Qx$ for some $x \in G$.

REMARK. If E is a commutative regular ring, then continuity, regularity, projectivity and injectivity are the same concept for ${}_E G$.

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Department of Mathematics
Seoul National University
Seoul 151–742, Korea