Comm. Korean Math. Soc. 7 (1992), No. 2, pp. 325–339

COMPATIBLE MAPPINGS OF TYPE (A) AND COMMON FIXED POINTS IN MENGER SPACES

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I. Introduction

K. Menger [18] introduced the notion of probabilistic metric spaces (or statistical metric spaces), which is a generalization of metric spaces, and the study of these spaces was expanded rapidly with the pioneering works of B. Schweizer and A. Sklar [20], [21]. Especially, the theory of probabilistic metric spaces is of fundamental importance in probabilistic functional analysis. For the detailed discussions of these spaces and their applications, we refer to [8], [9], [23], [31],[32], [25]-[27] and [38].

Recently, some fixed point theorems in probabilistic metric spaces have been proved by many authors; A.T. Bharucha-Reid [1], Gh. Bocsan [2], S.S. Chang [5], Lj.B. Ciric [6], O. Hadzic [10]-[12], T. L. Hicks [13], S.L. Singh [28]-[30], M. Stojakovic [33]-[35], N.X. Tan [36] and many others [3], [4], [7], [13], [19], [37]. Since every metric space is a probabilistic metric space, we can use many results in probabilistic metric spaces to prove some fixed point theorems in metric spaces and Banach spaces. Recently, G. Jungck [14] generalized the Banach contraction principle by using the concept of compatible mappings on metric spaces. Of course, any commuting mappings and weakly commuting are compatible mappings but the converses are not true [14]. The existence of fixed points for compatible mappings on metric spaces and probabilistic metric spaces is shown by G. Jungck [14]-[17], S.N. Mishra [19] and S. Sessa et al. [24].

In this paper, we introduce the concept of compatible mappings of type (A) on Menger spaces, which is equivalent to the concept of compatible mappings under some conditions. and prove some common fixed point theorems for compatible mappings of type (A) on Menger spaces and metric spaces. Our results extend, generalize and improve the results

Received March 5, 1992.

of A.T. Bharucha-Reid, G.L. Cain, Jr. and R.H. Kasriel, K.P. Chamola, R. Dedeic and N. Sarapa, S.N. Mishra, S.L. Singh and B.D. Pant, etc.

II. Preliminaries

Let R denote the set of reals and R^+ the nonnegative reals. A mapping $f : R \to R^+$ is called a distribution function if it is nondecreasing and left continuous with $\inf \mathcal{F} = 0$ and $\sup \mathcal{F} = 1$. We will denote $\mathcal{L}$ by the set of all distribution functions.

A probabilistic metric space (beriefly, a PM-space) is a pair $(X, \mathcal{F})$, where X is a nonempty set and F is a mapping from $X \times X$ to $\mathcal{L}$. For $(u, v) \in X \times X$, the distribution function $\mathcal{F}(u, v)$ is denoted by $F_{u,v}$. The functions $F_{u,v}$ are assumed to satisfy the following conditions:

(P1) $F_{u,v}(x) = 1$ for every $x > 0$ if and only if $u = v$,

(P2) $F_{u,v}(0) = 0$ for every $u, v \in X$,

(P3) $F_{v,u}(x) = F_{u,v}(x)$ for every $u, v \in X$,

(P4) If $F_{v,w}(x) = 1$ and $F_{u,w}(y) = 1$, then $F_{u,v}(x + y) = 1$ for every $u, v, w \in X$.

In a metric space (X, d), the metric d induces a mapping $F : X \times X \to \mathcal{L}$ such that

$$\mathcal{F}(u, v)(x) = F_{u,v}(x) = H(x - d(u, v))$$

for every $u, v \in X$ and $x \in R$, where H is a specific distribution function defined by

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0. \end{cases}$$

A function $t : [0,1] \times [0,1] \to [0,1]$ is called a T-norm if it satisfies the following conditions:

(t1) $t(a, 1) = a$ for every $a \in [0,1]$ and $t(0,0) = 0$,

(t2) $t(a, b) = t(b, a)$ for every $a, b \in [0,1]$,

(t3) If $c \geq a$ and $d \geq b$, then $t(c, d) \geq t(a, b)$,

(t4) $t(t(a, b), c) = t(a, t(b, c))$ for every $a, b, c \in [0,1]$.

A Menger space is a triple $(X, \mathcal{F}, t)$, where $(X, \mathcal{F})$ is a PM-space and t is a T-norm with the following condition:

(P5) $F_{v,w}(x+y) \geq t(F_{u,v}(x), F_{v,w}(y))$ for every $u, v, w \in X$ and $x, y \in R^+$.

The concept of neighbourhoods in PM-spaces was introduced by B. Schweizer and K. Sklar [20]. If $u \in X, \epsilon > 0$ and $\lambda \in (0,1)$, then an (ϵ, λ)-neighbourhood of u, denoted by $U_u(\epsilon, \lambda)$, is defined by

$$U_u(\epsilon, \lambda) = \{v \in X : F_{u,v}(\epsilon) > 1 - \lambda\}.$$

If $(X, \mathcal{F}, t)$ is a Menger space with the continuous T-norm t, then the family

$$\{U_u(\epsilon, \lambda) : u \in X, \epsilon > 0, \lambda \in (0,1)\}$$

of neighbourhoods induces a Hausdorff topology on X.

The following definitions and theorems are well-known [3],[22]:

DEFINITION 2.1. Let $(X, \mathcal{F}, t)$ be a Menger space. A mapping S from X into itself is said to be continuous at a point $p \in X$ if for every $\epsilon > 0$ and $\lambda > 0$, there exist $\epsilon_1 > 0$ and $\lambda_1 > 0$ such that if $q \in U_p(\epsilon_1, \lambda_1)$, then $Sq \in U_{Sp}(\epsilon, \lambda)$, that is, if $F_{p,q}(\epsilon_1) > 1 - \lambda_1$, then $F_{Sp,Sq}(\epsilon) > 1 - \lambda$.

DEFINITION 2.2. Let $(X, \mathcal{F}, t)$ be a Menger space with the continuous T-norm t. A sequence $\{p_n\}$ in X is said to be convergent to a point $p \in X$ if every $\epsilon > 0$ and $\lambda > 0$, there exists an integer $N = N(\epsilon, \lambda)$ such that $p_n \in U_p(\epsilon, \lambda)$ for all $n \geq N$, or equivalently, $F_{p,p_n}(\epsilon) > 1 - \lambda$ for all $n \geq N$. We write $p_n \to p$ as $n \to \infty$ or $\lim_{n\to\infty} p_n = p$.

Since the (ϵ, λ)-topology in a Menger space $(X, \mathcal{F}, t)$ satisfies the first axiom of the countability, we have the following:

THEOREM 2.1. *Let $(X, \mathcal{F}, t)$ be a Menger space with the continuous T-norm t and S be a mapping from X into itself. Then S is continuous at a point $p \in X$ if and only if for every sequence $\{p_n\}$ in X converging to p, the sequence $\{Sp_n\}$ converges to the point Sp.*

THEOREM 2.2. *Let $(X, \mathcal{F}, t)$ be Menger space with the continuous T-norm t. Then $\mathcal{F}$ is a lower semi-continuous function of points in X, that is, for any fixed $x \in R^+$, if $q_n \to q$ and $p_n \to p$ as $n \to \infty$, then*

$$\liminf_{n\to\infty} F_{p_n,q_n}(x) = F_{p,q}(x).$$

DEFINITION 2.3. Let $(X, \mathcal{F}, t)$ be a Menger space with the continuous T-norm t. A sequence $\{p_n\}$ in X is called a Cauchy sequence if for every $\epsilon > 0$ and $\lambda > 0$, there exists an integer $N = N(\epsilon, \lambda) > 0$ such that $F_{p_n, p_m}(\epsilon) > 1 - \lambda$ for all $m, n \geq N$.

DEFINITION 2.4. A Menger space $(X, \mathcal{F}, t)$ with the continuous T-norm t is said to be complete if every Cauchy sequence in X converges to a point in X.

The following theorems establish the relations between metric spaces and Menger spaces. Recall that the Menger space $(X, \mathcal{F}, t)$ induced by the metric d in a metric space (X, d) is called an induced Menger space.

THEOREM 2.3. *Let t be a T-norm defined by $t(a, b) = \min\{a, b\}$. Then the induced Menger space $(X, \mathcal{F}, t)$ is complete if a metric space (X, d) is complete.*

THEOREM 2.4. *Let $(X, \mathcal{F}, t)$ be an induced Menger space by the metric d in a metric space (X, d). Let $\{p_n\}$ be a sequence in X and S be a mapping from X into itself. Then for every $\epsilon > 0$ and $\lambda > 0$, $F_{p_n, p}(\epsilon) > 1 - \lambda$ if and only if there exists an integer N such that $d(p_n, p) < \epsilon$ for all $n \geq N$, and S is continuous at p in the sense of the Menger space if and only if S is continuous at p in the sense of the metric space.*

III. Compatible Mappings of Type (A)

In this section, motivated by the concept of compatible mappings on metric spaces and PM-spaces [14], [19], we introduce the concept of compatible mappings of type (A) on Menger spaces and show that the concepts of compatible mappings and compatible mappings of type (A) are equivalent under some conditions. In metric spaces, the concepts of compatible mappings and compatible mappings of type (A) are equivalent under some conditions [17].

DEFINITION 3.1. Let $(X, \mathcal{F}, t)$ be a Menger space such that the T-norm t is continuous and S, T be mappings from X into itself. S and T are said to be compatible if

$$\lim_{n \to \infty} F_{STx_n, TSx_n}(x) = 1$$

for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

DEFINITION 3.2. Let $(X, \mathcal{F}, t)$ be a Menger space such that the T-norm t is continuous and S, T be mappings from X into itself. S and T are said to be compatible of type (A) if

$$\lim_{n\to\infty} F_{TSx_n,SSx_n}(x) = 1 \text{ and } \lim_{n\to\infty} F_{STx_n,TTx_n}(x) = 1$$

for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

First, the following Propositions 3.1 and 3.2 show that Definitions 3.1 and 3.2 are equivalent under some conditions:

PROPOSITION 3.1. *Let $(X, \mathcal{F}, t)$ be a Menger space such that the T-norm t is continuous and $t(x,x) \geq x$ for all $x \in [0,1]$, and $S, T : X \to X$ be continuous mappings. If S and T are compatible, then they are compatible of type (A).*

Proof. Suppose that S and T are compatible. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$. Since S is continuous, we have

$$\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} STx_n = Sz$$

and so, for positive reals ϵ and λ, there exists an integer $M(\epsilon, \lambda)$ such that

$$F_{SSx_n,Sz}(\epsilon/2) > 1 - \lambda \text{ and } F_{STx_n,Sz}(\epsilon/2) > 1 - \lambda$$

for all $n \geq M(\epsilon, \lambda)$. Further, since S and T are compatible, we have

$$\lim_{n\to\infty} F_{STx_n,TSx_n}(\epsilon/2) = 1.$$

Therefore, from the following inequality:

$$F_{SSx_n,TSx_n}(\epsilon) \geq t(F_{SSx_n,STx_n}(\epsilon/2), F_{STx_n,TSx_n}(\epsilon/2)),$$

it follows that $\lim_{n\to\infty} F_{SSx_n,TSx_n}(\epsilon) = 1$. Similarly,

$$\lim_{n\to\infty} F_{TTx_n,STx_n}(\epsilon) = 1.$$

Therefore, S and T are compatible mappings of type (A). This completes the proof.

PROPOSITION 3.2. *Let $(X, \mathcal{F}, t)$ be a Menger space such that the T-norm t is continuous and $t(x,x) \geq x$ for all $x \in [0,1]$, and let $S, T : X \to X$ be compatible mappings of type (A). If one of S and T is continuous, then S and T are compatible.*

Proof. Assume, without loss of generality, that T is continuous. To show that S and T are compatible, suppose that $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$$

for some $z \in X$. Then $TSx_n, TTx_n \to Tz$ as $n \to \infty$ since T is continuous. By the condition (P5), we have

$$F_{STx_n,TSx_n}(\epsilon) \geq t(F_{STx_n,TTx_n}(\epsilon/2), F_{TTx_n,TSx_n}(\epsilon/2)). \tag{3.1}$$

Since S and T are compatible of type (A), we have

$$\lim_{n\to\infty} F_{TSx_n,SSx_n}(\epsilon/2) = 1 \text{ and } \lim_{n\to\infty} F_{STx_n,TTx_n}(\epsilon/2) = 1.$$

Moreover, $\lim_{n\to\infty} F_{TTx_n,TSx_n}(\epsilon/2) = F_{Tz,Tz}(\epsilon/2) = 1$. Therefore, from (3.1), it follows that $\lim_{n\to\infty} F_{STx_n,TSx_n}(\epsilon) = 1$. Therefore, S and T are compatible. This completes the proof.

The following proposition is a direct consequence of Propositions 3.1 and 3.1:

PROPOSITION 3.3. *Let $(X, \mathcal{F}, t)$ be a Menger space such that the T-norm t is continuous and $t(x,x) \geq x$ for all $x \in [0,1]$, and $S, T : X \to X$ be continuous mappings. Then S and T are compatible if and only if they are compatible of type (A).*

REMARK 1. In [17], we can find two examples that Proposition 3.3 is not true if S and T are not continuous on X.

Next, we give several properties of compatible mappings of type (A) on a Menger space for our main theorems:

PROPOSITION 3.4. *Let $(X, \mathcal{F}, t)$ be a Menger space such that the T-norm t is continuous and $t(x,x) \geq t$ for all $x \in [0,1]$, and $S,T : X \to X$ be mappings. If S and T are compatible of type (A) and $Sz = Tz$ for some $z \in X$, then $STz = TTz = TSz = SSz$.*

Proof. Suppose that $\{x_n\}$ is a sequence in X defined by $x_n = t, n = 1, 2, \cdots$, for some $z \in X$ and $Sz = Tz$. Then we have $Sx_n, Tx_n \to Sz$ as $n \to \infty$. Since S and T are compatible of type (A), for every $\epsilon > 0$,

$$F_{STz,TTz}(\epsilon) = \lim_{n\to\infty} F_{STx_n,TTx_n}(\epsilon) = 1.$$

Therefore, $STz = TTz$. Similarly, we have $TSz = SSz$. But $Tz = Sz$ implies $TTz = TSz$. Therefore, we have $STz = TTz = TSz = SSz$. This completes the proof.

PROPOSITION 3.5. *Let $(X, \mathcal{F}, t)$ be a Menger space such that the T-norm t is continuous and $t(x,x) \geq x$ for all $x \in [0,1]$, and $S,T : X \to X$ be mappings. Let S and T be compatible mappings of type (A) and $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$. Then we have*

(1) $\lim_{n\to\infty} TSx_n = Sz$ if S is continuous at z.

(2) $STz = TSz$ and $Sz = Tz$ if S and T are continuous at z.

Proof. (1) Suppose that S is continuous at z. Since $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$, we have $SSx_n \to Sz$ as $n \to \infty$, or equivalently, for any positive reals ϵ and λ, there exists an integer $M(\epsilon, \lambda)$ such that $F_{SSx_n,Sz}(\epsilon/2) > 1 - \lambda$ for all $n \geq M(\epsilon, \lambda)$. Since S and T are compatible of type (A), for every $\epsilon > 0$, $\lim_{n\to\infty} F_{TSx_n,SSx_n}(\epsilon) = 1$ and so we have

$$F_{TSx_n,Sz}(\epsilon) \geq t(F_{TSx_n,SSx_n}(\epsilon/2), F_{SSx_n,Sz}(\epsilon/2)) > 1 - \lambda$$

for all $n \geq M(\epsilon, \lambda)$, which means that $TSx_n \to Sz$ as $n \to \infty$.

(2) Suppose that $S, T : X \to X$ are continuous at z. Since $Tx_n \to z$ as $n \to \infty$ and S is continuous at z, by Proposition 3.5 (1), $TSx_n \to Sz$ as $n \to \infty$. On the other hand, since $Sx_n \to z$ as $n \to \infty$ and T is also continuous at z, $TSx_n \to Tz$. Thus, we have $Sz = Tz$ by the uniqueness of the limit and so, by Proposition 3.4, $STz = TTz = TSz = SSz$. Therefore, $TSz = STz$. This completes the proof.

IV. Common Fixed Point Theorems

Before proving our main theorems, we need the following lemma [29]:

LEMMA 4.1. *Let $\{x_n\}$ be a sequence in a Manger space $(X, \mathcal{F}, t)$, where t is a continous T-norm and $t(x, x) \geq x$ for all $x \in [0, 1]$. If there exists a constant $k \in (0, 1)$ such that*

$$F_{x_n, x_{n+1}}(kx) \geq F_{x_{n-1}, x_n}(x)$$

for all $x > 0$ and $n \in N$, then $\{x_n\}$ is a Cauchy sequence in X.

REMARK 2. In Propositions 3.1, 3.5 and Lemma 4.1, the conditions "the T-norm t is continuous and $t(x, x) \geq x$ for all $x \in [0, 1]$" can be replaced by "$t(x, y) = \min\{x, y\}$ for all $x, y \in [0, 1]$". In fact, since $t(a, 1) = a$ and $t(1, b) = b$ for all $a, b \in [0, 1]$, we have

$$t(a, b) \leq \min\{t(a, 1), t(1, b)\} = \min\{a, b\}$$

for all $a, b \in [0, 1]$. On the other hand, we have

$$t(a, b) \geq t(\min\{a, b\}, \min\{a, b\}) \geq \min\{a, b\}$$

for all $a, b \in [0, 1]$, which implies $t(a, b) = \min\{a, b\}$.

Now, we are ready to prove our main theorems :

THEOREM 4.2. *Let $(X, \mathcal{F}, t)$ be a complete Menger space with $t(x, y) = \min\{x, y\}$ for all $x, y \in [0, 1]$ and A, B, S, T be mappings from X into itself such that*
(4.1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,
(4.2) the pairs A, S and B, T are compatible of type (A),

(4.3) one of A, B, S and T is continuous,

(4.4) there exists a constant $k \in (0,1)$such that

$$
\begin{aligned}
F_{Au,Bv}(kx) \geq \min\{&(F_{Su,Tv}(x))^2, F_{Su,Au}(x)F_{Tv,Bv}(x),\\
&F_{Su,Bv}(2x)F_{Tv,Au}(x), F_{Tv,Au}(x),\\
&F_{Su,Bv}(2x)F_{Tv,Bv}(x)\}
\end{aligned}
$$

for all $u, v \in X$ and $x \geq 0$. Then A, B, S and T have a unique common fixed point in X.

Proof. By (4.1), since $A(X) \subset T(X)$, for any arbitrary $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subset S(X)$, for this point x_1, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

$$
\begin{cases} y_{2n} = Tx_{2n+1} = Ax_{2n} \\ y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \end{cases} \tag{4.5}
$$

for $n = 0, 1, 2, \cdots$. Now, we prove $F_{y_{2n},y_{2n+1}}(kx) \geq F_{y_{2n-1},y_{2n}}(x)$ for all $x > 0$, where $k \in (0,1)$. Suppose that $F_{y_{2n},y_{2n+1}}(kx) < F_{y_{2n-1},y_{2n}}(x)$ for some $x > 0$. Then by using (4.4) and $F_{y_{2n},y_{2n+1}}(kx) \geq F_{y_{2n},y_{2+1}}(x)$, we have

$$\begin{aligned}
(F_{y_{2n},y_{2n+1}}(kx))^2 &\geq (F_{Ax_{2n},Bx_{2n+1}}(kx))^2 \\
&> \min\{(F_{Sx_{2n},Tx_{2n+1}}(x))^2, \\
&\quad F_{Sx_{2n},Ax_{2n}}(x)F_{Tx_{2n+1},Bx_{2n+1}}(x), \\
&\quad F_{Sx_{2n},Bx_{2n+1}}(2x)F_{Tx_{2n+1},Ax_{2n}}(x), \\
&\quad F_{Sx_{2n},Ax_{2n}}(x)F_{Tx_{2n+1},Ax_{2n}}(x), \\
&\quad F_{Sx_{2n},Bx_{2n+1}}(2x)F_{Tx_{2n+1},Bx_{2n+1}}(x)\} \\
&= \min\{(F_{y_{2n-1},y_{2n}}(x))^2, F_{y_{2n-1},y_{2n}}(x)F_{y_{2n},y_{2n+1}}(x), \\
&\quad F_{y_{2n-1},y_{2n+1}}(2x)F_{y_{2n},y_{2n}}(x), \\
&\quad F_{y_{2n-1},y_{2n}}(x)F_{y_{2n},y_{2n}}(x), \\
&\quad F_{y_{2n-1},y_{2n+1}}(x)F_{y_{2n},y_{2n+1}}(x)\} \\
&> \min\{(F_{y_{2n-1},y_{2n}}(x))^2, F_{y_{2n-1},y_{2n}}(x)F_{y_{2n},y_{2n+1}}(x), \\
&\quad t(F_{y_{2n-1},y_{2n}}(x), F_{y_{2n},y_{2n+1}}(x)), F_{y_{2n-1},y_{2n}}(x), \\
&\quad t(F_{y_{2n-1},y_{2n}}(x), F_{y_{2n},y_{2n+1}}(x))F_{y_{2n},y_{2n+1}}(x)\} \\
&> \min\{(F_{y_{2n},y_{2n+1}}(kx))^2, F_{y_{2n},y_{2n+1}}(kx))^2 \\
&\quad F_{y_{2n},y_{2n+1}}(kx), F_{y_{2n},y_{2n+1}}(kx), (F_{y_{2n},y_{2n+1}}(kx))^2\} \\
&= (F_{y_{2n},y_{2n+1}}(kx))^2,
\end{aligned}$$

which is a contradiction. Thus, we have $F_{y_{2n},y_{2n+1}}(kx) \geq F_{y_{2n-1},y_{2n}}(x)$. Similarly, we have also $F_{y_{2n+1},y_{2n+2}}(kx) \geq F_{y_{2n},y_{2n+1}}(x)$. Therefore, for every $n \in N, F_{y_n,y_{n+1}}(kx) \geq F_{y_{n-1},y_n}(x)$. Therefore, by Lemma 4.1, $\{y_n\}$ is a Cauchy sequence in X. Since the Menger space $(X, \mathcal{F}, t)$ is complete, $\{y_n\}$ converges to a point z in X and the subsequences $\{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Sx_{2n}\}, \{Tx_{2n+1}\}$ of $\{y_n\}$ also converge to z.

Now, suppose that T is continuous. Since B and T are compatible of type (A), by Proposition 3.5, $BTx_{2n+1}, TTx_{2n+1} \to Tz$ as $n \to \infty$. Putting $u = x_{2n}$ and $v = Tx_{2n+1}$ in (4.4), we have

$$(4.6)\qquad \begin{aligned}(F_{Ax_{2n},BTx_{2n+1}}(kx))^2 \geq \min\{&(F_{Sx_{2n},TTx_{2n+1}}(x))^2,\\ &F_{Sx_{2n},Ax_{2n}}(x)F_{TTx_{2n+1},BTx_{2n+1}}(x),\\ &F_{Sx_{2n},BTx_{2n+1}}(2x)F_{TTx_{2n+1},Ax_{2n}}(x),\\ &F_{Sx_{2n},Ax_{2n}}(x)F_{TTx_{2n},Ax_{2n}}(x),\\ &F_{Sx_{2n},BTx_{2n+1}}(2x)F_{TTx_{2n+1},BTx_{2n+1}}(x)\}.\end{aligned}$$

Taking $n \to \infty$ in (4.6), we have

$$\begin{aligned}(F_{z,Tz}(kx))^2 &\geq \min\{(F_{z,Tz}(x))^2, F_{Tz,Tz}(x), F_{z,Tz}(2x)F_{Tz,z}(x),\\ &\qquad F_{z,z}(x)F_{Tz,z}(x), F_{z,Tz}(2x)F_{Tz,Tz}(x)\}\\ &= (F_{z,Tz}(x))^2,\end{aligned}$$

which implies that $Tz = z$. Again, replacing u by x_{2n} and v by z in (4.4), we have

(4.7)

$$\begin{aligned}(F_{Ax_{2n},Bz}(kx))^2 \geq \min\{&(F_{Sx_{2n},Tz}(x))^2, F_{Sx_{2n},Ax_{2n}}(x)F_{Tz,Bz}(x),\\ &F_{Sx_{2n},Bz}(2x)F_{Tz,Ax_{2n}}(x), F_{Sx_{2n},Ax_{2n}}(x)F_{Tz,Ax_{2n}}(x),\\ &F_{Sx_{2n},Bz}(2x)F_{Tz,Bz}(x)\}.\end{aligned}$$

Taking $n \to \infty$ in (4.7), we have

$$\begin{aligned}(F_{z,Bz}(kx))^2 &\geq \min\{(F_{z,Tz}(x))^2, F_{z,Bz}(x), F_{z,Bz}(2x)F_{Tz,z}(x),\\ &\qquad F_{z,z}(x)F_{Tz,z}(x), F_{z,Bz}(2x)F_{z,Bz}(x)\}\\ &= (F_{z,Bz}(x))^2,\end{aligned}$$

which implies that $Bz = z$. Since $B(X) \subset S(X)$, there exists a point w in X such that $Bz = Sw = z$. By using (4.4) again, we have

$$\begin{aligned}(F_{Aw,z}(kx))^2 &= (F_{Aw,Bz}(kx))^2\\ &\geq \min\{(F_{Sw,Tz}(x))^2, F_{Sw,Aw}(x)F_{Tz,Bz}(x)\\ &\qquad F_{Sw,Bz}(2x)F_{Tz,Aw}(x), F_{Sw,Aw}(x)F_{Tz,Aw}(x),\\ &\qquad F_{Sw,Bz}(2x)F_{Tz,Bz}(x)\}\\ &= (F_{Aw,z}(x))^2,\end{aligned}$$

which means that $Aw = z$. Since A and S are compatible of type (A) and $Aw = Sw = z$, by Proposition 3.4, we have, for every $\epsilon > 0$, $F_{ASw,SSw}(\epsilon) = 1$ and so $Az = ASw = SSw = Sz$. By using (4.4) again, we have $Az = z$. Therefore, $Az = Bz = Sz = Tz = z$, that is, z is a common fixed point of the given mappings. The uniqueness of the common fixed point z follows easily from (4.4). This completes the proof.

As a consequence of Theorems 2.3 and 4.2, we have the following:

THEOREM 4.3. *Let A, B, S and T be mappings from a complete metric space (X, d) into itself such that*
(4.8) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,
(4.9) one of A, B, S and T is continuous,
(4.10) the pairs A, S and B, T are compatible of type (A),
(4.11) there exists a constant $k \in (0,1)$ such that

$$(d(Ax, By))^2 \le k \max\{(d(Sx, Ty)^2, d(Sx, Ax)d(Ty, By), \frac{1}{2}d(Sx, By)d(Ty, Ax), d(Sx, Ax)d(Ty, Ax), \frac{1}{2}d(Sx, By)d(Ty, By)\}$$

for all x, y in X. Then A, B, S and T have a unique common fixed point in X.

ACKNOWLEDGEMENT. The authors are indebted to Prof. S. S. Chang and Prof. S. N. Mishra for their careful readings of the manuscript and their suggestions.

References

[1] A.T. Bharucha-Reid, *Fixed Point Theorems in Probabilistic analysis*, Bull. Amer. Math. Soc. **82** (1976), 641–657.

[2] Gh. Bocsan, *On some fixed Point Theorems in Probabilistic Metric Spaces*, Math. Balkanica 4 (1974), 67–70.

[3] G.L. Cain, Jr. and R.H. Kasriel, *Fixed and Periodic Points of Local Contraction Mappings on Probabilistic Metric Spaces*, Math. Systems Theory **9** (1976), 289–297.

[4] K.P. Chamola, *Fixed Points of Mappings satisfying a new Contraction Condition in Random Normed Spaces*, Math. Japon. **33(6)** (1988), 821–825.

[5] S.S. Chang, *On Some Fixed Point Theorems in PM-spaces and its Applications*, Z. Wahrsch. verw. Gebiete **63** (1983), 463–474.

[6] Lj. B. Ciric, *On Fixed Points of Generalized Contractions on Probabilistic Metric Spaces*, Publ. Inst. Math. Beograd **18(32)** (1975), 71–78.

[7] R. Dedeic and N. Sarapa, *A Common Fixed Point Theorem for Three Mappings on Menger Spaces*, Math. Japon. **34(6)** (1989), 919–923.

[8] R.J. Egbert, *Products and Quotients of Probabilistic Metric Spaces*, Pacific J. Math. **24** (1968), 437–455.

[9] O. Hadzic, *On the (ϵ, λ)-topology of LPC-spaces*, Glasnik Mat. **13(33)** (1978), 293–297.

[10] O. Hadzic, *A Fixed Point Theorem in Probabilistic Locally Convex Spaces*, Rev. Roum. Math. Pures Appl. **23** (1978), 735–744.

[11] O. Hadzic, *A Fixed Point Theorem in Menger Spaces*, Publ. Inst. Math. Beograd **20(40)** (1979), 107–112.

[12] O. Hadzic, *Some Theorems on the Fixed Points in Probabilistic Metric and Random Normed Spaces*, Boll. Un. Mat. Ital. **13(5)** **18** (1981), 1–11.

[13] T.L. Hicks, *Fixed Point Theory in Probabilistic Metric Spaces*, Review of Research, Fac. Sci., Math. Series, Univ. of Novi Sad. **13** (1983), 63–72.

[14] G. Jungck, *Compatible Mappings and Common Fixed Points*, Internat. J. Math. and Math. Sci. **9(4)** (1986), 771–779.

[15] G. Jungck, *Compatible Mappings and Common Fixed Points (2)*, Internat. J. Math. and Math. Sci. **11(2)** (1988), 285–288.

[16] G. Jungck, *Common Fixed Points for Commuting and Compatible Maps on Compacta*, Proc. Amer. Math. Soc. **103** (1988), 977–983.

[17] G. Jungck, P.P. Marthy and Y. J. Cho, *Compatible Mappings of Type (A)*, to appear in Math. Japon.

[18] K. Menger, *Statistical Metrics*, Proc. Nat. Acad. Sci., U.S.A. **28** (1942), 535–537.

[19] S. N. Mishra, *Common Fixed Points of Compatible Mappings in PM-spaces*, Math. Japon. **36(2)** (1991), 283–289.

[20] B. Schweizer and A. Sklar, *Statistical Metric Spaces*, Pacific J. Math. **10** (1960), 313–334.

[21] B. Schweizer and A. Sklar, *Probabilistic Metric Spaces, Vol 5*, North-Holland Series in Probability and Applied Math, 1983.

[22] V.Sehgal and A.T. Rharnucha-Reid, *Fixed Points of Contraction Mappings on Probabilistic Metric Spaces*, Math. Systems Theory **6** (1972), 72–102.

[23] A. N. Serstnev, *The Notion of Random Normed Spaces*, Dokl. Akad. Nauk. USSR **149** (1963), 280–283.

[24] S. Sessa, B.E. Rhoades and M.S. Khan, *On Common Fixed Points of Compatible Mappings in Metric and Banach Spaces*, Internat. J. Math. and Math. Sci. **11(2)** (1988), 375–392.

[25] H. Sherwood, *On the Completion of Probabilistic Metric Spaces*, Z. Wahrsch. verw. Gebiete. **6** (1966), 62–64.

[26] H. Sherwood, *On E-spaces and Their Relation to Other Classes of Probabilistic Metric Spaces*, J. London Math. Soc. **44** (1969), 441–448.

[27] H. Sherwood, *Complete Probabilistic Metric Spaces*, Z. Wahrsch. verw. Gebiete **20** (1971), 117–128.

[28] S.L. Singh and B.D. Pant, *Fixed Point Theorems for Commuting Mappings in Probabilistic Metric Spaces*, Honam Math. J. **5** (1985), 139–150.

[29] S.L. Singh and B.D. Pant, *Common Fixed Point Theorems in Probabilistic Metric Spaces and Extension to Uniform Spaces*, Honam Math. J. **6** (1984), 1–12.

[30] S.L. Singh and B.D. Pant, *Coincidence and Fixed Point Theorems for a Family of Mappings on Menger Spaces and Extension to Uniform Spaces*, Math. Japon. **33(6)** (1988), 957–973.

[31] A. Spacek, *Note on K. Menger's Probabilistic Geometry*, Czechoslovak Math. J. **6** (1956), 72–74.

[32] R.R. Stevens, *Metrically generated Probabilistic Metric Spaces*, Fund. Math. **61** (1968), 259–269.

[33] M. Stojakovic, *Fixed Point Theorem in Probabilistic Metric Spaces*, Kobe J. Math. **2** (1985), 1–9.

[34] M. Stojakovic, *Common Fixed Point Theorems in Complete Metric and Probabilistic Metric Spaces*, Bull. Austral. Math. Soc. **36** (1987), 73–88.

[35] M. Stojakovic, *a Common Fixed Point Theorems in Probabilistic Meric Spaces and its Applications*, Glasnik Mat. **23(43)** (1988), 203–211.

[36] N.X. Tan, *Generalized Probabilistic Metric Spaces and Fixed Point Theorems*, Math. Nachr. **129** (1986), 205–218.

[37] R. Vasuki, *A Fixed Point Theorem for a Sequence of Maps satisfying a new Contraction Type Condition in Menger Spaces*, Math. Japon. **36(6)** (1990), 1099–1102.

[38] A. Wald, *On a Statistical Generalization of Metric Spaces*, Proc. Nat. Acad. Sci., U.S.A. **29** (1943), 196–197.

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