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FINITE RANK PERTURBATIONS OF SEMI–FREDHOLM OPERATORS

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The "finite rank perturbation theorem"of Fredholm theory says that if X and Y are Banach spaces, if $T \in BL(X,Y)$ is semi-Fredholm, and if $K \in BL(X,Y)$ is finite rank then $T+K$ is semi Fredholm([1] Corollary V.2.2; [2] Theorem 6.12.2). In this note we extend this result to incomplete normed spaces.

We recall [2] that if X and Y are normed spaces then if $k > 0$ and if $\|x\| \leq k\|Tx\|$ for each $x \in X$ then we call $T \in BL(X,Y)$ ***bounded below***, if T is bounded below and has a closed range then we call T ***closed***, if $y \in \{Tx : \|x\| \leq k\|Tx\|\}$ for each $y \in Y$ then we call T ***open*** and if $y \in cl\{Tx : \|x\| \leq k\|Tx\|\}$ ***almost open***. The operator $T \in BL(X,Y)$ will be called ***relatively open*** (respectively, ***relatively almost open***) if its truncation $T^\wedge : X \longrightarrow T(X)$ is open(respectively, almost open)(cf. [1], [4]). Thus bounded below is just relatively open one-one. Relative openness can be tested with the (***reduced***) ***minimum modulus***

$$\gamma(T) = \inf\{\|Tx\| : \mathrm{dist}(x, T^{-1}(0)) \geq 1\} \quad \text{if } 0 \neq T \in BL(X,Y):$$

if $T = 0$ we may take $\gamma(T) = \infty$. Evidently, T is relatively open if and only if $\gamma(T) > 0$ (cf.[3]). If X and Y are complete then T is relatively open if and only if T has a closed range (cf. [1] Theorem IV.1.6). If M and N are normed spaces we shall assume $M \times N$ is a normed space in such a way as to have the Cartesian product topology. If M and N are closed subspaces of X satisfying

$$M + N = X \quad \text{and} \quad M \cap N = \{0\} \tag{0.1}$$

then there is a one-one correspondence

$$m + n \longleftrightarrow (m+n) : X \longleftrightarrow M \times N. \tag{0.2}$$

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We shall say that M and N are *complemented* subspaces if they satisfy (0.1), while the topology of X is the same as that induced on it by the mapping (0.2) and the Cartesian topology of the product $M \times N$. To indicate that this is so we shall write $X = M \oplus N$.

To prove the main result we need to three lemmas.

LEMMA 1. *If $T \in BL(X,Y)$ is relatively open and if M is a subspace of X then the restriction of T to $M + T^{-1}(0)$ is relatively open.*

Proof. If T_1 is the restriction of T to $M + T^{-1}(0)$ then $T_1^{-1}(0) = T^{-1}(0)$. Hence $\gamma(T_1) \geq \gamma(T) > 0$, which says that T_1 is relatively open.

LEMMA 2. *Let $T \in BL(X,Y)$. If M is a closed subspace of X with dim $X/M < \infty$ and if T_M is the restriction of T to M then*

$$T_M \text{ closed} \Longrightarrow T \text{ relatively open.}$$

Proof. Suppose M is a closed subspace of X with dim $X/M < \infty$. If T_M, the restriction of T to M, is closed then there is a finite dimensional subspace N of X for which

$$X = M \oplus N \quad \text{and} \quad T^{-1}(0) \subseteq N,$$

and thus

$$T(X) = T(M) \oplus T(N),$$

where $T(M)$ and $T(N)$ are both closed. We thus have ([2] Theorem 2.5.1)

$$X \simeq M \times N \quad \text{and} \quad T(X) \simeq T(M) \times T(N).$$

Since, in particular, N is finite dimensional it follows from the open mapping theorem that T_N, the restriction of T to N, is relatively open. Therefore, for each $x = y + z \in X$ with $y \in M$ and $z \in N$, we have

$$\begin{aligned} \|Tx\| = \|T_M(y) + T_N(z)\| &\geq k(\|T_M(y)\| + \|T_N(z)\|) \\ &\geq k(k'\|y\| + k''\|z'\|) \quad \text{with } z' \in z + T^{-1}(0) \\ &\geq K\|y + z'\| \quad \text{with } K = \inf\{kk', kk''\}, \end{aligned}$$

which says that T is relatively open.

LEMMA 3. *Let $T \in BL(X,Y)$. Suppose that for any closed subspace M of X with $\dim X/M < \infty$, T_M is not bounded below. Then given $\epsilon > 0$, there is an infinite dimensional subspace $M(\epsilon)$ of X such that T restricted to $M(\epsilon)$ has norm not exceeding ϵ.*

Proof. See [1] Theorem III.1.9.

We recall ([2, Definition 3.2.7]) that $T \in BL(X,Y)$ is ***proper*** if the mapping
$\operatorname{core}(T){:}X/T^{-1}(0) \longrightarrow \operatorname{cl}(TX)$ defined by setting

$$\operatorname{core}(T)(x + T^{-1}(0)) = Tx \in \operatorname{cl}(TX) \quad \text{for each } x \in X$$

is invertible. Evidently,

$$T \text{ proper} \Longleftrightarrow T \text{ relatively open and } T(X) \text{ closed}$$

We also recall ([2] Definition 6.10.1) that the operator $T \in BL(X,Y)$ is called *upper semi-Fredholm* if it is proper with finite dimensional null space, and *lower semi-Fredholm* if it is proper with the closure of its range of finite co-dimension. If T is either upper or lower semi-Fredholm we shall call it *semi-Fredholm*, and *Fredholm* if it is both. These concepts are also ([2] (6.12.1.19)) dual to one another:

$$T \text{ lower semi-Fredholm} \Longleftrightarrow T^* \quad \text{upper semi-Fredholm.}$$

The *index* of a Fredholm operator is given by

$$\operatorname{index}(T) = \dim T^{-1}(0) - \dim Y/T(X).$$

It is known ([2] Theorem 6.10.2) that the index is continuous even in the incomplete spaces.

We are ready for finite rank perturbations of the semi-Fredholm operators for incomplete spaces. The third part of the next theorem was noticed in [2, Theorem 6,3,4].

THEOREM 1. *If T and K are in $BL(X,Y)$ then*

$$(1.1)\quad \begin{aligned}&T \ \textit{upper semi-Fredholm}, \quad K \ \textit{finite rank}\\ &\qquad\qquad \Longrightarrow T+K \ \textit{upper semi-Fredholm},\end{aligned}$$

$$(1.2)\quad \begin{aligned}&T \ \textit{lower semi-Fredholm}, \quad K \ \textit{finite rank}\\ &\qquad\qquad \Longrightarrow T+K \ \textit{lower semi-Fredholm},\end{aligned}$$

and hence

$$(1.3)\quad \begin{aligned}&T \ \textit{Fredholm}, \quad K \ \textit{finite rank}\\ &\Longrightarrow T+K \ \textit{Fredholm with } \mathit{index}(T+K)=\mathit{index}(T).\end{aligned}$$

Proof. Suppose $T \in BL(X,Y)$ is upper semi-Fredholm. Since $T^{-1}(0)$ is finite dimensional, we can find a closed subspace M of X for which

$$X = M \oplus T^{-1}(0).$$

Let T_M be the restriction of T to M. Then T_M is bounded below and hence there is $k > 0$ for which

$$k\|x\| \leq \|T_M(x)\| \quad \text{for each } x \in M.$$

Suppose $K \in BL(X,Y)$ is finite rank. Clearly, $T+K$ has a closed range because $T(X)$ is closed and $K(X)$ is finite dimensional. Assume $T+K$ is not relatively open. Thus, if K_M is the restriction of K to M then, by Lemma 2, $T_M + K_M$ is not closed. Since $T_M(X) = T(X)$ is closed and $K_M(X)$ is finite dimensional, it follows that $T_M + K_M$ has a closed range and thus that $T_M + K_M$ is not bounded below. Hence, by Lemma 3, there is an infinite dimensional subspace M_0 of M for which

$$\|(T_M + K_M)x\| \leq \frac{k}{2}\|x\| \quad \text{for all } x \in M_0.$$

It therefore follows that

$$\|K_M(x)\| \geq \|T_M(x)\| - \|(T_M + K_M)x\| \geq \frac{k}{2}\|x\| \quad \text{for all } x \in M_0,$$

which says that the restriction of K to M_0 is bounded below. This, however, contradicts the fact that K is finite rank. Therefore $T+K$ is relatively open; thus $T+K$ is proper. We now prove that $(T+K)^{-1}(0)$ is finite dimensional. There is a closed subspace N_1 for which

$$(1.4) \qquad (T+K)^{-1}(0) = \{(T+K)^{-1}(0) \cap T^{-1}(0)\} \oplus N_1.$$

Then, by Lemma 1, the restriction of T to $N_1+T^{-1}(0)$ is relatively open, and thus the restriction of T to N_1 is bounded below. Since $K=-T$ on N_1 and K is finite rank, N_1 must be finite dimensional. Thus, by (1.4), $(T+K)^{-1}(0)$ is finite dimensional. This proves (1.1). For (1.2) apply (1.1) to the duals of T and K. Towards (1.3) suppose $T \in BL(X,Y)$ is Fredholm. Since λK is finite rank for each scalar λ, we have, by (1.1) and (1.2), $T+\lambda K$ is also Fredholm. Further, by the continuity of the index ([2] (6.10.2.3)), index$(T+\lambda K)$ is a continuous function of λ.

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