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ON THE INTEGRABILITY OF A ηK–CONFORMAL KILLING EQUATION IN COSYMPLECTIC MANIFOLDS

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1. Introduction

Let M^n be an n-dimensional Riemannian manifold. Denote respectively by $g_{ab}, R_{abcd}, R_{ab} = R_{rab}{}^r$ and $R = R_{ab}g^{ab}$ the metric, the curvature tensor, the Ricci tensor, and the scalar curvature of Riemannian manifold in terms of local coordinates $\{x^a\}$, where Latin indices run over the range $\{1, 2, \cdots, n\}$.

In M^n, a p-form $u (p \geq 1)$ is said to be *Killing* ([5], [6]) if it satisfies

$$\nabla_b u_{a_1 a_2 \cdots a_p} + \nabla_{a_1} u_{b a_2 \cdots a_p} = 0,$$

which is called the Killing-Yano's equation. ∇ denotes the operator of covariant differentiation.

The following theorem is well known.

THEOREM A ([2], [8]). *A necessary and sufficient condition in order that the Killing-Yano's equation is completely integrable is that the Riemannian manifold $M^n (n > 2)$ is a space of constant curvature.*

In a Sasakian manifold M^n, a 1-form u is called *D-Killing of type* α [9] if it satisfies the following equation

$$\nabla_a u_b + \nabla_b u_a = -2\alpha u_r(\phi_a{}^r \eta_b + \phi_b{}^r \eta_a),$$

where α is constant. We call this is the D-Killing equation of type α.

Then the following theorem is well known.

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THEOREM B ([9]). *A necessary and sufficient condition in order that the D-Killing equation of type α is completely integrable is that the Sasakian manifold $M^n(n > 3)$ is a space of constant ϕ-holomorphic sectional curvature with $H = 1 - 4\alpha$.*

The purpose of this paper is to consider the analogy of Theorem A and B in cosymplectic manifold. That is, we will prove the followings.

THEOREM 3.1. *If there exists (locally) a horizontal ηK-conformal Killing 2-form u with its associated 1-form ρ satisfying $u(X,Y)(P) = C(X,Y)$ for any vector fields X and Y perpendicular to η and point P of a cosymplectic manifold $M^n(n > 5)$ and any skew symmetric constants C, then M^n is a space of constant ϕ-holomorphic sectional curvature.*

THEOREM 4.1. *A necessary and sufficient condition in order that the horizontal ηK-conformal Killing equation of Theorem 3.1 is completely integrable is that the cosymplectic manifold $M^n(n > 5)$ is a space of constant ϕ-holomorphic sectional curvature.*

In section 2, we give some fundamental formulas in cosymplectic manifolds to fix our notations and introduce some operators and the proof of Theorem 3.1 will be given in section 3.

Moreover, we denote ourselves to prove the integrability condition of the ηK-conformal Killing equation in section 4.

2. Preliminaries

We represent tensors by their components with respect to the natural basis and use the summation convention. For a differential p-form

$$u = \frac{1}{p!} u_{a_1 \cdots a_p} dx^{a_1} \wedge \cdots \wedge dx^{a_p}$$

with skew symmetric coefficients $u_{a_1 \cdots a_p}$, the coefficients of its exterior differential du and the exterior codifferential δu are given respectively by

$$(du)_{a_1 \cdots a_{p+1}} = \sum_{i=1}^{p+1} (-1)^{i+1} \nabla_{a_i} u_{a_1 \cdots \hat{a}_i \cdots a_{p+1}},$$

$$(\delta u)_{a_2 \cdots a_p} = -\nabla^r u_{r a_2 \cdots a_p},$$

where $\nabla^r = g^{rs}\nabla_s$, and $\hat{a}_i$ means a_i to be deleted.

A cosymplectic manifold M^n with metric g is that M^n admitting a parallel tensor field $\phi_a{}^b$ and a parallel vector field ξ^a such that

$$
\begin{aligned}
&\phi_a{}^r\phi_r{}^b = -\delta_a{}^b + \eta_a{}^{\xi^b},\ \phi_a{}^r\xi^a = 0,\\
&\eta_a\phi_b^a = 0, \eta_a\xi^a = 1,
\end{aligned}
\tag{2.1}
$$

where we put $\phi_{ab} = \phi_a{}^r g_{rb}$.

By the Ricci's identity, the followings are well known in a cosymplectic manifold.

$$
R_{abr}{}^c\eta^r = 0, R_{ar}\eta^r = 0.
\tag{2.2}
$$

Moreover, we know the following equations in a cosymplectic manifold.

$$
\begin{aligned}
&R_{abr}{}^c\phi_d{}^r = R_{abd}{}^r\phi_r{}^c,\\
&R_{ar}\phi_b{}^r = -R_{rabs}\phi^{rs},\\
&R_{ar}\phi_b{}^r = -R_{br}\phi_a{}^r = \frac{1}{2}R_{abrs}\phi^{rs},
\end{aligned}
\tag{2.3}
$$

where we put $R_{ar}\phi_b{}^r = S_{ba} = -S_{ab}$.

In the sequel, we consider a cosymplectic manifold M^n and assume that $n > 2$.

Now we want to recall some operators for differential forms in M^n. Denote by F^p the set of all p-forms on M^n. The operators $\Gamma : F^p \longrightarrow F^{p+1}, \Phi : F^p \longrightarrow F^p$ and the inner product $i(\eta) : F^p \longrightarrow F^{p-1}$ of 1-form η are defined respectively by

$$
\begin{aligned}
(\Gamma u)_{a_0\cdots a_p} &= \sum_{i=0}^{p}(-1)^i\phi_{a_i}{}^r\nabla_r u_{a_0\cdots\hat{a}_i\cdots a_p},\\
(\Phi u)_{a_1\cdots a_p} &= \sum_{i=1}^{p}\phi_{a_i}{}^r u_{a_1\cdots r\cdots a_p},\\
(i(\eta)u)_{a_2\cdots a_p} &= \eta^r u_{ra_2\cdots a_p}
\end{aligned}
$$

for any p-form $u(p \geq 1)$. For 0-form u_0, we define $\Phi u_0 = 0$ and $i(\eta)u_0 = 0$.

If $i(\eta)u$ vanishes identically, then a p-form u is said to be ***horizontal.***

In the present paper, we put $\gamma_{ab} = g_{ab} - \eta_a \eta_b$. It is well known in [10] that γ_{ab} is positive definite for any vector $X^a \neq (\eta_r X^r)\eta^a$.

If the Ricci tensor of M^n is of the form $R_{ab} = \alpha\gamma_{ab}$ for some function α, then M^n is called a *cosymplectic η-Einstein manifold* [4]. By contraction, we have $\alpha = \dfrac{R}{n-1}$, where R is the scalar curvature of M^n. Thus a cosymplectic η-Einstien manifold is characterized by $R_{ab} = \dfrac{R}{n-1}\gamma_{ab}$.

A cosymplectic manifold is called of ***constant ϕ-holomorphic sectional*** curvature if the *curvature* tensor satisfies the following equation:

$$R_{abc}{}^r = \frac{R}{(n-1)(n+1)}(\gamma_{bc}\gamma_a{}^r - \gamma_{ac}\gamma_b{}^r + \phi_{bc}\phi_a{}^r - \phi_{ac}\phi_b{}^r - 2\phi_{ab}\phi_c{}^r). \tag{2.4}$$

A 2-form u is said to be *ηK-conformal Killing*, if there exists a associated 1-form ρ such that

$$\nabla_b u_{cd} + \nabla_c u_{bd} = 2\rho_d\gamma_{bc} - \rho_b\gamma_{cd} - \rho_c\gamma_{bd} + 3(\tilde{\rho}_b\phi_{cd} + \tilde{\rho}_c\phi_{bd}), \tag{2.5}$$

where, we put

$$\rho_a = \frac{1}{n+1}[2(i\rho)\eta_a - (\delta u)_a],$$
$$\tilde{\rho}_a = \phi_a{}^r\rho_r = (\Phi\rho)_a \text{ and } i\rho = i(\eta)\rho = \eta^a\rho_a.$$

On the other hand, we proved the following [4].

THEOREM C. *Let M be a cosymplectic manifold of dimension n admitting a ηK-conformal Killing 2-form u. If u is horizontal, then the associated 1-form ρ of u is also horizontal.*

Thus, here and in the sequel, we assume that

$$i(\eta)u = iu = 0. \tag{2.6}$$

Then, it is easy to see from tranvecting (2.5) with $\phi_a{}^c$ that

$$\phi_a{}^r\nabla_r u_{bd} + \nabla_a(\Phi u)_{bd} = (\Gamma u)_{abd} - 2\rho_a\phi_{bd} - \rho_b\phi_{ad} + \rho_d\phi_{ab} - \tilde{\rho}_b\gamma_{ad} + \tilde{\rho}_d\gamma_{ab}, \tag{2.7}$$

where we used theorem C.

By interchanging alternatively indices as $a \to b \to d$ at (2.7) and adding all together, we find

$$d\Phi u = 2\Gamma u. \tag{2.8}$$

3. A sufficient condition for M to be a space of constant ϕ-holomorphic sectional curvature

Hereafter, we deal with a ηK-conformal Killing 2-form u_{ab} of (2.6). At first, we have the following equation ([4]) :

$$\begin{aligned} & R_{bcd}{}^{r}u_{ar} + R_{adc}{}^{r}u_{br} + R_{dab}{}^{r}u_{cr} + R_{cba}{}^{r}u_{dr} \\ & \quad = -\sigma_{bd}\gamma_{ca} + \sigma_{dc}\gamma_{ba} + \sigma_{ab}\gamma_{cd} - \sigma_{ac}\gamma_{bd} \\ & \qquad + \tilde{\sigma}_{ab}\phi_{cd} + \tilde{\sigma}_{ca}\phi_{bd} + \tilde{\sigma}_{cd}\phi_{ab} + \tilde{\sigma}_{db}\phi_{ac} \\ & \qquad + 2(\tilde{\sigma}_{da}\phi_{bc} + 2\tilde{\sigma}_{cb}\phi_{ad}), \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} \sigma_{ab} &= \rho_{ab} + \rho_{ba} = \nabla_{a}\rho_{b} + \nabla_{b}\rho_{a} \\ &= \frac{1}{(n-3)(n+3)}[n(R_{a}{}^{r}u_{br} + R_{b}{}^{r}u_{ar}) \\ &\quad - 3(R_{c}{}^{r}u_{dr} + R_{d}{}^{r}u_{cr})\phi_{a}{}^{d}\phi_{b}{}^{c}], \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \tilde{\sigma}_{ab} &= \tilde{\rho}_{ab} - \tilde{\rho}_{ba} = \nabla_{a}\tilde{\rho}_{b} + \nabla_{b}\tilde{\rho}_{a} = (d\Phi\rho)_{ab} \\ &= \frac{n+2}{(n+1)(n+3)}(R_{b}{}^{r}\phi_{r}{}^{e}u_{ae} - R_{a}{}^{r}\phi_{r}{}^{e}u_{be}) \\ &\quad - \frac{1}{(n+1)(n+3)}(\phi_{b}{}^{r}R_{a}{}^{e} - \phi_{a}{}^{r}R_{b}{}^{e})u_{re}. \end{aligned} \tag{3.3}$$

By virtue of (3.2) and (3.3), the equation (3.1) is rewritten as

(3.4)

$$
\begin{aligned}
&R_{bcd}{}^{r}u_{ar}+R_{adc}{}^{r}u_{br}+R_{dab}{}^{r}u_{cr}+R_{cba}{}^{r}u_{dr}\\
&=\frac{1}{(n-3)(n+3)}[n(R_{a}{}^{r}u_{br}+R_{b}{}^{r}u_{ar})\gamma_{cd}\\
&\quad-3(\phi_{a}{}^{r}S_{b}{}^{s}+\phi_{b}{}^{r}S_{a}{}^{s})u_{rs}\gamma_{cd}-n(R_{a}{}^{r}u_{cr}+R_{c}{}^{r}u_{ar})\gamma_{bd}\\
&\quad+3(\phi_{a}{}^{r}S_{c}{}^{s}+\phi_{c}{}^{r}S_{a}{}^{s})u_{rs}\gamma_{bd}+n(R_{d}{}^{r}u_{cr}+R_{c}{}^{r}u_{dr})\gamma_{ab}\\
&\quad-3(\phi_{c}{}^{r}S_{d}{}^{s}+\phi_{d}{}^{r}S_{c}{}^{s})u_{rs}\gamma_{ab}-n(R_{b}{}^{r}u_{dr}+R_{d}{}^{r}u_{br})\gamma_{ac}\\
&\quad+3(\phi_{b}{}^{r}S_{d}{}^{s}+\phi_{d}{}^{r}S_{b}{}^{s})u_{rs}\gamma_{ac}]\\
&\quad+\frac{1}{(n+j)(n+3)}[(n+2)(S_{b}{}^{r}u_{ar}-S_{a}{}^{r}u_{br})\phi_{cd}\\
&\quad-(\phi_{a}{}^{r}R_{c}{}^{s}-\phi_{c}{}^{r}R_{a}{}^{s})u_{rs}\phi_{bd}+(n+2)(S_{d}{}^{r}u_{cr}-S_{c}{}^{r}u_{dr})\phi_{ab}\\
&\quad+(\phi_{c}{}^{r}R_{d}{}^{s}-\phi_{d}{}^{r}R_{c}{}^{s})u_{rs}\phi_{ab}-(n+2)(S_{d}{}^{r}u_{br}-S_{b}{}^{r}u_{dr})\phi_{ac}\\
&\quad-(\phi_{b}{}^{r}R_{d}{}^{s}-\phi_{d}{}^{r}R_{b}{}^{s})u_{rs}\phi_{ac}+2(n+2)(S_{b}^{r}u_{cr}-S_{c}{}^{r}u_{br})\phi_{ad}\\
&\quad+2(\phi_{c}{}^{r}R_{b}{}^{s}-\phi_{b}{}^{r}R_{c}{}^{s})u_{rs}\phi_{ad}-2(n+2)(S_{d}{}^{r}u_{ar}-S_{a}{}^{r}u_{dr})\phi_{bc}\\
&\quad-2(\phi_{a}{}^{r}R_{d}{}^{s}-\phi_{d}{}^{r}R_{a}{}^{s})u_{rs}\phi_{bc}].
\end{aligned}
$$

Under the assumption of Theorem 3.1, the skew symmetric part of coefficients of u_{rs} in (3.4) vanish, so we have from (2.6) and (3.4)

$$
\begin{aligned}
&R_{bcd}{}^{r}\gamma_{a}{}^{s}+R_{adc}{}^{r}\gamma_{b}{}^{s}+R_{dab}{}^{r}\gamma_{c}{}^{s}+R_{cba}{}^{r}\gamma_{d}{}^{s}\\
&\qquad-R_{bcd}{}^{s}\gamma_{a}{}^{r}-R_{adc}{}^{s}\gamma_{b}{}^{r}-R_{dab}{}^{s}\gamma_{c}{}^{r}-R_{cba}{}^{s}\gamma_{d}{}^{r}\\
&=\frac{1}{(n-3)(n+3)}\\
&\quad[n(R_{a}{}^{r}\gamma_{b}{}^{s}+R_{b}{}^{r}\gamma_{a}{}^{s}-R_{a}{}^{s}\gamma_{b}{}^{r}-R_{b}{}^{s}\gamma_{a}{}^{r})\gamma_{cd}\\
&\quad+3(\phi_{a}{}^{r}S_{b}{}^{s}+\phi_{b}{}^{r}S_{a}{}^{s}-\phi_{a}{}^{s}S_{b}{}^{r}-\phi_{b}{}^{s}S_{a}{}^{r})\gamma_{cd}\\
&\quad-n(R_{a}{}^{r}\gamma_{c}{}^{s}+R_{c}{}^{r}\gamma_{a}{}^{s}-R_{a}{}^{s}\gamma_{c}{}^{r}-R_{c}{}^{s}\gamma_{a}{}^{r})\gamma_{bd}\\
&\quad-3(\phi_{a}{}^{r}S_{c}{}^{s}+\phi_{c}{}^{r}S_{a}{}^{s}-\phi_{a}{}^{s}S_{c}{}^{r}-\phi_{c}{}^{s}S_{a}{}^{r})\gamma_{bd}\\
&\quad+n(R_{c}{}^{r}\gamma_{d}{}^{s}+R_{d}{}^{r}\gamma_{c}{}^{s}-R_{c}{}^{s}\gamma_{d}{}^{r}-R_{d}{}^{s}\gamma_{c}{}^{r})\gamma_{ab}
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
&+ 3(\phi_c{}^r S_d{}^s + \phi_d{}^r S_c{}^s - \phi_c{}^s S_d{}^r - \phi_d{}^s S_c{}^r)\gamma_{ab} \\
&- n(R_b{}^r \gamma_d{}^s + R_d{}^r \gamma_b{}^s - R_b{}^s \gamma_d{}^r - R_d{}^s \gamma_b{}^r)\gamma_{ac} \\
&- 3(\phi_b{}^r S_d{}^s + \phi_d{}^r S_b{}^s - \phi_b{}^s S_d{}^r - \phi_d{}^s S_b{}^r)\gamma_{ac}] \\
-&\frac{1}{(n+j)(n+3)} \\
&[(n+2)(S_a{}^r \gamma_b{}^s - S_b{}^r \gamma_a{}^s - S_a{}^s \gamma_b{}^r + S_b{}^s \gamma_a{}^r)\phi_{cd} \\
&+ (\phi_a{}^r R_b{}^s - \phi_b{}^r R_a{}^s - \phi_a{}^s R_b{}^r + \phi_b{}^s R_a{}^r)\phi_{cd} \\
&- (n+2)(S_a{}^r \gamma_c{}^s - S_c{}^r \gamma_a{}^s - S_a{}^s \gamma_c{}^r + S_c{}^s \gamma_a{}^r)\phi_{bd} \\
&- (\phi_a{}^r R_c{}^s + \phi_c{}^r R_a{}^s - \phi_a{}^s R_c{}^r + \phi_c{}^s R_a{}^r)\phi_{bd} \\
&+ (n+2)(S_c{}^r \gamma_d{}^s - S_d{}^r \gamma_c{}^s - S_c{}^s \gamma_d{}^r + S_d{}^s \gamma_c{}^r)\phi_{ab} \\
&+ (\phi_c{}^r R_d{}^s - \phi_d{}^r R_c{}^s - \phi_c{}^s R_d{}^r + \phi_d{}^s R_c{}^r)\phi_{ab} \\
&- (n+2)(S_b{}^r \gamma_d{}^s - S_d{}^r \gamma_b{}^s - S_b{}^s \gamma_d{}^r + S_d{}^s \gamma_b{}^r)\phi_{ac} \\
&- (\phi_b{}^r R_d{}^s - \phi_d{}^r R_b{}^s - \phi_b{}^s R_d{}^r - \phi_d{}^s R_b{}^r)\phi_{ac} \\
&+ 2(n+2)(S_c{}^r \gamma_b{}^s - S_b{}^r \gamma_c{}^s - S_c{}^s \gamma_b{}^r + S_b{}^s \gamma_c{}^r)\phi_{ad} \\
&+ 2(\phi_c{}^r R_b{}^s - \phi_b{}^r R_c{}^s - \phi_c{}^s R_b{}^r + \phi_b{}^s R_c{}^r)\phi_{ad} \\
&- 2(n+2)(S_a{}^r \gamma_d{}^s - S_d{}^r \gamma_a{}^s - S_a{}^s \gamma_d{}^r + S_d{}^s \gamma_a{}^r)\phi_{bc} \\
&- 2(\phi_a{}^r R_d{}^s - \phi_d{}^r R_a{}^s - \phi_a{}^s R_d{}^r + \phi_d{}^s R_a{}^r)\phi_{bc}].
\end{aligned}
$$

Contracting (3.5) on s and d and making use of the first Bianchi's identity, (2.2) and (2.3) we obtain

$$
\begin{aligned}
&(n-2)R_{cba}{}^r = \frac{1}{(n+1)(n^2-9)} \\
&[(n^3 - n^2 - 5n + 9)(\gamma_{ab} R_c{}^r - \gamma_{ac} R_b{}^r + R_{ab}\gamma_c{}^r - R_{ac}\gamma_b{}^r) \\
&+ 4n(\phi_c{}^r S_{ba} + \phi_b{}^r S_{ac} + 2\phi_a{}^r S_{bc}) \\
&+ (n^3 - 3n^2 - 9n + 15)(\phi_{ba} S_c{}^r + \phi_{ac} S_b{}^r + 2\phi_{bc} S_a{}^r) \\
&+ (n-3)R(\phi_{ba}\phi_c{}^r + \phi_{ac}\phi_b{}^r + 2\phi_{bc}\phi_a{}^r) \\
&+ n(n+1)R(\gamma_{ac}\gamma_b{}^r - \gamma_{ab}\gamma_c{}^r)].
\end{aligned}
\tag{3.6}
$$

Again, transvecting (3.6) with $\phi_e{}^c\phi_t{}^b$ we have

$$\begin{aligned}(n-2)R_{cba}{}^{r} = {} & \frac{1}{(n+1)(n^2-9)}\\ & [(n^3-n^2-5n+9)(\phi_{ba}S_c{}^r - \phi_{ca}S_b{}^r + S_{ba}\phi_c{}^r - S_{ca}\phi_b{}^r)\\ & + 4n(R_{ba}\gamma_c{}^r - R_{ca}\gamma_b{}^r + 2S_{cb}\phi_a{}^r)\\ & + (n^3-3n^2-9n+15)(\gamma_{ba}R_c{}^r - \gamma_{ca}R_b{}^r - 2\phi_{cb}S_a{}^r)\\ & + (n-3)R(\gamma_{ab}\gamma_c{}^r - \gamma_{ca}\gamma_b{}^r - 2\phi_{cb}\phi_a{}^r)\\ & + n(n+1)R(\phi_{ca}\phi_b{}^r - \phi_{ba}\phi_c{}^r)].\end{aligned}\tag{3.7}$$

where we used (2.3). Thus, it follows from (3.6) and (3.7) that

$$\begin{aligned}&(n^3-n^2-5n+9)(\gamma_{ba}R_c{}^s - \gamma_{ca}R_b{}^s + R_{ba}\gamma_c{}^s - R_{ca}\gamma_b{}^s\\ &- \phi_{ba}S_c{}^s + \phi_{ca}S_b{}^s - S_{ba}\phi_c{}^s + S_{ca}\phi_b{}^s)\\ &+ (n^3-3n^2-9n+15)(\phi_{ba}S_c{}^s - \phi_{ca}S_b{}^s - 2\phi_{cb}S_a{}^s - \gamma_{ba}R_c{}^s\\ &+ \gamma_{ca}R_b{}^s + 2\phi_{cb}S_a{}^s)\\ &+ 4n(S_{ba}\phi_c{}^s - S_{ca}\phi_b{}^s - 2S_{cb}\phi_a{}^s - R_{ba}\gamma_c{}^s + R_{ca}\gamma_b{}^s + 2S_{cb}\phi_a{}^s)\\ &- n(n+1)R(\gamma_{ba}\gamma_c{}^s - \gamma_{ca}\gamma_b{}^s - \phi_{ba}\phi_c{}^s + \phi_{ca}\phi_b{}^s)\\ &+ (n-3)R(\phi_{ba}\phi_c{}^s - \phi_{ca}\phi_b{}^s - 2\phi_{cb}\phi_a{}^s - \gamma_{ba}\gamma_c{}^s + \gamma_{ca}\gamma_b{}^s\\ &+ 2\phi_{cb}\phi_a{}^s) = 0.\end{aligned}\tag{3.8}$$

By contraction (3.8) with $\delta_s{}^c$, we can obtain

$$R_{ab} = \frac{R}{n-1}\gamma_{ab}(n > 5),\tag{3.9}$$

which means that the cosymplectic manifold $M^n(n > 5)$ is η-Einsteinnian. Substituting (3.9) into (3.6), we can easily find that the curvature tensor is the form of (2.4). This means that $M^n(n > 5)$ is a space of constant ϕ-holomorphic sectional curvature. Consequently, we complete the proof of Theorem 3.1.

4. Integrability of a ηK-conformal Killing equation[1)]

The pupose of this section is to prove Theorem 4.1, that is, we will show that the converse of Theorem 3.1 is true. In this section, we assume that (2.6) holds good. First of all, we will prepare the followings.

THEOREM D([4]). *If a cosymplectic manifold $M^n (n > 3)$ is an η-Einstein manifold, then the non-parallel horizontal associated* 1*-form ρ_a of ηK-conformal Killing* 2*-form is Killing, that is,*

$$\nabla_a \rho_b + \nabla_b \rho_a = 0.$$

Moreover, by virtue of (3.3) we have the following.

LEMMA 4.1. *If a cosymplectic manifold M is an η-Einstein manifold, then we have for the horizontal associated* 1*-form ρ of horizontal ηK-conformal Killing* 2*-form u*

$$d\Phi\rho = \frac{R}{(n-1)(n+1)}\Phi u,$$

hence, Φu is closed 2-form if $R \neq 0$.

In a cosymplectic manifold M^n we consider the ηK-conformal Killing equation as a system of partial differential equations of unknown function u_{ab}. This system is equivalent to the following system of unknown functions $u_{ab}(= -u_{ba})$ and $u_{abc}(= -u_{acb})$:

$$u_{bcd} + u_{cbd} = 2\rho_d \gamma_{bc} - \rho_b \gamma_{cd} - \rho_c \gamma_{bd} + 3(\tilde{\rho}_b \phi_{cd} + \tilde{\rho}_c \phi_{bd}), \tag{4.1}$$

$$\nabla_b u_{cd} = u_{bcd}, \tag{4.2}$$

[1)] In this section, we assume that M and all quantities are real analytic.

$$
\begin{aligned}
\nabla_a u_{bcd} =& \frac{1}{2}(R_{dca}{}^r u_{br} + R_{bda}{}^r u_{cr} + R_{cba}{}^r u_{dr}) \\
&+ \frac{1}{2}(\rho_{bd} - \rho_{db})\gamma_{ca} + \frac{1}{2}(\rho_{cb} - \rho_{bc})\gamma_{ad} \\
&+ \frac{1}{2}(\rho_{dc} - \rho_{cd})\gamma_{ab} - \rho_{ac}\gamma_{bd} + \rho_{ad}\gamma_{bc} \\
&+ \frac{1}{2}(\tilde{\rho}_{bc} - \tilde{\rho}_{cb})\phi_{ad} + \frac{1}{2}(\tilde{\rho}_{cd} - \tilde{\rho}_{dc})\phi_{ab} \\
&+ \frac{1}{2}(\tilde{\rho}_{db} - \tilde{\rho}_{bd})\phi_{ac} + \frac{1}{2}(\tilde{\rho}_{da} - \tilde{\rho}_{ad})\phi_{bc} \\
&+ (\tilde{\rho}_{ac} - \tilde{\rho}_{ca})\phi_{bd} + (2\tilde{\rho}_{ab} + \tilde{\rho}_{ba})\phi_{cd}.
\end{aligned}
\tag{4.3}
$$

From now on, we will show the above system is completely integrable if $M^n (n > 5)$ is a space of constant ϕ-holomorphic sectional curvature.

From our assumption, the equation (4.3) is rewritten as

$$
\begin{aligned}
\nabla_a u_{bcd} =& \frac{R}{(n-1)(n+1)}[\gamma_{ab} u_{dc} + \gamma_{ac} u_{bd} + \gamma_{ad} u_{cb} \\
&+ \phi_{ab}(\Phi u)_{cd} + \phi_{ac}(\Phi u)_{db} + \phi_{ad}(\Phi u)_{bc} \\
&- \phi_{cd}(\Phi u)_{ab} - \phi_{db}(\Phi u)_{ac} - \phi_{bc}(\Phi u)_{ad} \\
&- (\phi_{cb} u_{dr} + \phi_{bd} u_{cr} + \phi_{dc} u_{br})\phi_a{}^r] \\
&+ \gamma_{ba}\rho_{dc} + \gamma_{ca}\rho_{bd} + \gamma_{da}\rho_{cb} + \gamma_{bd}\rho_{ca} \\
&+ \gamma_{bc}\rho_{ad} + 3\phi_{cd}\tilde{\rho}_{ab},
\end{aligned}
\tag{4.4}
$$

where we used Theorem D, Lemma 4.1 and (2.6).

The equation obtained from (4.1) by differentiation:

$$
\begin{aligned}
\partial_a u_{bcd} + \partial_a u_{cbd} = \partial_a [& 2\rho_d \gamma_{bc} - \rho_b \gamma_{cd} - \rho_c \gamma_{bd} \\
&+ 3(\tilde{\rho}_b \phi_{cd} + \tilde{\rho}_c \phi_{bd})]
\end{aligned}
$$

is satisfied identically by (4.1), (4.2) and (4.4).

Next, we will discuss the integrability condition of (4.2).

$$
\nabla_a \nabla_b u_{cd} - \nabla_b \nabla_a u_{cd} = -R_{abc}{}^r u_{rd} - R_{abd}{}^r u_{cr}. \tag{4.5}
$$

Taking account that M^n is a space of constant ϕ-holomorphic sectional curvature, we find from (2.4)

$$\begin{aligned}-R_{abc}{}^{r}u_{rd} - R_{abd}{}^{r}u_{cr} =& \frac{R}{(n-1)(n+1)}[\gamma_{ac}u_{bd} - \gamma_{bc}u_{ad} + \gamma_{ad}u_{cb} - \gamma_{bd}u_{ca} \\ &+ \phi_{b}{}^{r}(\phi_{ac}u_{rd} + \phi_{ad}u_{cr}) - \phi_{a}{}^{r}(\phi_{bc}u_{rd} + \phi_{bd}u_{cr}) \\ &+ 2\phi_{ab}(\Phi u)_{cd}].\end{aligned}$$

On the other hand, by virtue of (4.2), (4.4) and Lemma 4.1, the left hand side of (4.5) is equal to the right hand side of the above equation. Thus (4.5) holds good.

Finally, we will discuss the integrability condition of (4.4) as

$$(4.6)\quad \nabla_{e}\nabla_{a}u_{bcd} - \nabla_{a}\nabla_{e}u_{bcd} = -R_{eab}{}^{r}u_{rcd} - R_{eac}{}^{r}u_{brd} - R_{Ead}{}^{r}u_{bcr}.$$

Since, M^n is a space of constant ϕ-holomorphic sectional curvature, we can obtain from (2.4)

$$(4.7)\quad \begin{aligned}&-R_{eab}{}^{r}u_{rcd} - R_{eac}{}^{r}u_{brd} - R_{ead}{}^{r}u_{bcr}. \\ &\quad = \frac{R}{(n-1)(n+1)}[\gamma_{eb}u_{acd} - \gamma_{ab}u_{ccd} + \gamma_{ec}u_{bad} - \gamma_{ac}u_{bed} \\ &\quad + \gamma_{ed}u_{bca} - \gamma_{ad}u_{bce} + (\phi_{eb}u_{rcd} + \phi_{ec}u_{brd} + \phi_{ed}u_{bcr})\phi_{a}{}^{r} \\ &\quad - (\phi_{ab}u_{rcd} + \phi_{ac}u_{brd} + \phi_{ad}u_{bcr})\phi_{e}{}^{r} \\ &\quad + 2(\phi_{b}{}^{r}u_{rcd} + \phi_{c}{}^{r}u_{brd} + \phi_{d}{}^{r}u_{bcr})\phi_{ea}],\end{aligned}$$

where we used (2.6).

On the other hand, operating ∇_e to (4.4) and making use of (4.1), (4.2) and Theorem C we obtain

$$\begin{aligned}\nabla_e\nabla_a u_{bcd} = & \frac{R}{(n-1)(n+1)}[-\gamma_{ab}u_{ecd} - \gamma_{ac}u_{bed} - \gamma_{ad}u_{bce} \\ & + \gamma_{ac}(2\rho_d\gamma_{be} - \rho_b\gamma_{ed} - \rho_e\gamma_{bd} + 3\tilde{\rho}_b\phi_{ed}) \\ & - \gamma_{ad}(2\rho_c\gamma_{be} - \rho_b\gamma_{ec} - \rho_e\gamma_{bc} + 3\tilde{\rho}_b\phi_{ec}) \\ & + \nabla_e(\phi_{ab}(\Phi u)_{cd} + \rho_{ac}(\Phi u)_{db} + \phi_{ad}(\Phi u)_{bc}) \\ & - \nabla_e(\phi_{cd}(\Phi u)_{ab} + \phi_{db}(\Phi u)_{ac} + \phi_{bc}(\Phi u)_{ad}) \\ & + (\phi_{bc}u_{red} + \phi_{db}u_{rec} + \phi_{cd}u_{reb})\phi_a{}^r \\ & + \phi_{bc}(3\rho_a\phi_{ed} - 2\rho_d\phi_{ae} + \rho_e\phi_{ad} - \tilde{\rho}_a\gamma_{ed}) \\ & + \phi_{db}(3\rho_a\phi_{ec} - 2\rho_c\phi_{ae} + \rho_e\phi_{ac} + \tilde{\rho}_a\gamma_{ec}) \\ & + \phi_{cd}(3\rho_a\phi_{eb} - 2\rho_b\phi_{ae} + \rho_e\phi_{ab} + \tilde{\rho}_a\gamma_{eb} + 3\tilde{\rho}_e\gamma_{ab})] \\ & + \nabla_e(\gamma_{ba}\rho_{dc} + \gamma_{ca}\rho_{bd} + \gamma_{da}\rho_{cb} + \gamma_{bd}\rho_{ca} + \gamma_{bc}\rho_{ad}) \\ & + 3\phi_{cd}\nabla_e\tilde{\rho}_{ab}.\end{aligned}$$

By interchanging the indices e and a in the above equation, subtracting it from the original one and owing to Theorem C, (2.7), (2.8), (4.1), (4.2), Theorem D, Lemma 4.1 and the Ricci's identity, we can find that $\nabla_e\nabla_a u_{bcd} - \nabla_a\nabla_e u_{bcd}$ is reduced to the right hand side of (4.7). Therefore (4.6) holds good. Consequently, we complete the proof of Theorem 4.1.

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