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WEIERSTRASS SEMIGROUPS OF PAIRS OF POINTS ON ALGEBRAIC CURVES

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1. Introduction

Let C be a nonsingular complex projective curve (or a compact Riemann surface) of genus g. For a divisor D on C, dim $|D|$ means the dimension of the complete linear series containing D, which is the same as the projective dimension of the vector space of meromorphic functions on C with the pole divisor $(f)_\infty \le D$. The symbol g^r_n denotes a linear series of dimension r, of degree n.

Let $M(C)$ denote the field of meromorphic functions on C. For a pair (p, q) of points on C, we define the Weierstrass semigroup of (p, q) by

$$H(p,q) = \{(\alpha, \beta) \in N \times N | \text{ there exists } f \in M(C) \text{ with } (f)_\infty = \alpha p + \beta q\},$$

where N denotes the set of all non-negative integers. Indeed, this set forms a sub-semigroup of $N \times N$ [2].

In this paper, we give some properties about Weierstrass semigroups. In particular, we study about Weierstrass semigroups on trigonal curves.

We use the following notations:

$$N_{\le\alpha} = \{n \in N | n \le \alpha\},$$
$$N_{\ge\alpha} = \{n \in N | n \ge \alpha\}.$$

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Using the Riemann-Roch theorem, for each α and β with $\alpha+\beta \geq 2g$, $N_{\geq \alpha} \times N_{\geq \beta}$ is contained in $H(p,q)$.

We give the partial order on the set $N \times N$ as follows:

$$(\alpha,\beta) \leq (\gamma,\delta) \text{ if and only if } \alpha \leq \gamma \text{ and } \beta \leq \delta.$$

2. Some properties of Weierstrass semigroups of pairs

In this section, we obtain some properties of a Weierstrass semigroup of pair of points on an arbitrary curve of genus g.

THEOREM 2.1. *If p and q are distinct points on C, then the Weierstrass semigroup $H(p,q)$ contains the least upper bound of any two elements in it.*

Proof: Let (α_1,β_1) and (α_2,β_2) be elements in $H(p,q)$. We may assume that $\alpha_1 \geq \alpha_2$ and $\beta_1 \leq \beta_2$. Then (α_1,β_2) is the least upper bound of two elements. Since the linear series $|\alpha_1 p + \beta_1 q|$ is base point free and

$$|\alpha_1 p + \beta_2 q| = ||\alpha_1 p + \beta_1 q| + (\beta_2 - \beta_1)q|,$$

p is not a base point of $|\alpha_1 p + \beta_2 q|$. Similarly, q is not a base point of $|\alpha_1 p + \beta_2 q|$. Thus $|\alpha_1 p + \beta_2 q|$ is base point free and hence (α_1,β_2) is an element of $H(p,q)$.

THEOREM 2.2. *Let $(\alpha,\beta) \in H(p,q)$. If $\beta \leq \beta'$ and $\dim|\alpha p + \beta' q| = \dim|\alpha p + (\beta'-1)q| + 1$, then $(\alpha,\beta') \in H(p,q)$. Similarly, if $\alpha \leq \alpha'$ and $\dim|\alpha' p + \beta q| = \dim|(\alpha'-1)p + \beta q| + 1$, then $(\alpha',\beta) \in H(p,q)$.*

Proof: It suffices to show the first statement. Since $|\alpha p + \beta q|$ is base point free and

$$|\alpha p + \beta' q| = ||\alpha p + \beta q| + (\beta' - \beta)q|,$$

p is not a base point of the linear series $|\alpha p + \beta' q|$. And since $\dim|\alpha p + \beta' q| = \dim|\alpha p + (\beta' - 1)q| + 1$, q is not a base point of $|\alpha p + \beta' q|$. Thus $|\alpha p + \beta' q|$ is a base point free linear series and hence $(\alpha, \beta') \in H(p, q)$.

THEOREM 2.3. *For each element* $(\alpha, \beta) \in H(p, q)$, *there exists a totally ordered subset of* $H(p, q)$,

$$\{(\alpha_0, \beta_0), (\alpha_1, \beta_1), \cdots, (\alpha_{r-1}, \beta_{r-1}), (\alpha_r, \beta_r), (\alpha_{r+1}, \beta_{r+1}), \cdots\},$$

satisfying that $\dim|\alpha_i p + \beta_i q| = i$ *and* $(\alpha_r, \beta_r) = (\alpha, \beta)$.

Proof: For each $i \in N$, let n_i be the least number such that

$$\dim|\alpha p + (\beta + n_i)q| = r + i.$$

Then $\dim|\alpha p + (\beta + n_i)q| = \dim|\alpha p + (\beta + n_i - 1)q| + 1$. Thus $(\alpha, \beta + n_i)$ is in $H(p, q)$ by Theorem 2.2.

On the other hand, choose an element $(\alpha_{r-1}, \beta_{r-1})$ in $N \times N$ satisfying that

$$\dim|\alpha_{r-1}p + \beta_{r-1}q| = r - 1$$

and

$$\alpha_{r-1} + \beta_{r-1} = \min\{\gamma + \delta \,|\, \dim|\gamma p + \delta q| = r - 1, \gamma \leq \alpha, \delta \leq \beta\}.$$

Such an element does exist, since the dimension of the linear series decreases as the coefficients of p and q decreases. Then by the choice of $(\alpha_{r-1}, \beta_{r-1})$, we obtain

$$\begin{aligned} \dim|\alpha_{r-1}p + \beta_{r-1}q| &= \dim|(\alpha_{r-1} - 1)p + \beta_{r-1}q| + 1 \\ &= \dim|\alpha_{r-1}p + (\beta_{r-1} - 1)q| + 1. \end{aligned}$$

Thus neither p nor q is a base point of $|\alpha_{r-1}p + \beta_{r-1}q|$. Hence

$$(\alpha_{r-1}, \beta_{r-1}) \in H(p, q).$$

Repeating this process, we obtain a desired sequence.

THEOREM 2.4. *Let* $(m,n) \in N \times N$ *and* $\dim|mp+nq| = r$. *Then there exists an element* $(\alpha,\beta) \in H(p,q)$ *such that*

$$\alpha \leq m, \beta \leq n \text{ and } \dim|\alpha p + \beta q| = r.$$

Proof: Varying the element (γ,δ) in $N_{\leq m} \times N_{\leq n}$, $\dim|\gamma p+\delta q|$ admits every integer from 0 to r. Choose an element (α,β) in $N_{\leq m} \times N_{\leq n}$ satisfying the following two conditions

$$\dim|\alpha p + \beta q| = r,$$

and

$$\alpha + \beta = \min\{\gamma+\delta \mid \gamma \leq m, \delta \leq n, \dim|\gamma p + \delta q| = r\}.$$

Then by the choice of (α,β),

$$\begin{aligned}\dim|\alpha p + \beta q| &= \dim|(\alpha-1)p + \beta q| + 1 \\ &= \dim|\alpha p + (\beta-1)q| + 1.\end{aligned}$$

Thus (α,β) is contained in $H(p,q)$.

3. Weierstrass semigroups of pairs on a trigonal curve

Let C be a trigonal curve of genus $g \geq 5$. Recall that there exists a unique linear series g_3^1 on C. We say that C is of the n-th kind if $\dim|ng_3^1| = n$ and $\dim|(n+1)g_3^1| \geq n+2$. We know that the number n satisfies the inequality $(g-1)/3 \leq n \leq g/2$ [3]. If we let $m = g-n-1$, then m is the number satisfying that

$$mg_3^1 \text{ is special, but } (m+1)g_3^1 \text{ is not special.}$$

We say that a linear series g_k^r is compounded of g_3^1 if $g_n^r = rg_3^1 + B$, where B is the base locus of g_n^r. We can find the following lemma in [1].

LEMMA 3.1. *For any special linear series g_k^r on a trigonal curve C, g_k^r or its residual series is compounded of g_3^1.*

THEOREM 3.2. *Let p and q be points on a trigonal curve of genus g. If $(\alpha, \beta) \in H(p,q)$ and $\dim |\alpha p + \beta q| = 1$, then*

$$\alpha + \beta = 3 \text{ or } \alpha + \beta \geq \frac{(g+2)}{2}.$$

Proof: If the linear series $|\alpha p + \beta q|$ is nonspecial then, by Riemann-Roch theorem, $\dim |\alpha p + \beta q| = \alpha + \beta - g$. Thus $\alpha + \beta = g + 1$, which is larger than $(g+2)/2$.

If $|\alpha p + \beta q|$ is special, then $|\alpha p + \beta q|$ or $K - (\alpha p + \beta q)$ is compounded of g_3^1 by above lemma, where K is the canonical series on C.

If $|\alpha p + \beta q|$ is compounded of g_3^1, then $|\alpha p + \beta q| = g_3^1$, since $|\alpha p + \beta q|$ is a base point free linear series of dimension 1.

Now let $K - (\alpha p + \beta q)$ be compounded of g_3^1. By Brill-Nöther reciprocity,

$$\begin{aligned} K - (\alpha p + \beta q) &= g_{2g-2-\alpha-\beta}^{g-\alpha-\beta} \\ &= (g - \alpha - \beta) g_3^1 + B, \end{aligned}$$

where B is the base locus of $K - (\alpha p + \beta q)$. Thus

$$2g - 2 - \alpha - \beta \geq 3(g - \alpha - \beta),$$

and hence

$$\alpha + \beta \geq \frac{g+2}{2}.$$

THEMOREM 3.3. *Let C be a trigonal curve of n-th kind. Let p and q be points on C such that $2p+q \in g_3^1$ and $p \neq q$. Then each element (α, β) of $H(p,q)$ in $N_{\leq 2n} \times N_{\leq n}$ is of the form $(2k,k)$ for some $k = 0, 1, \cdots, n$.*

Proof: Let (α, β) be an element in $H(p,q)$ with $\alpha \leq 2n$ and $\beta \leq n$. Then $|\alpha p + \beta q|$ is a subseries of $ng_3^1 = |2np + nq|$. Since ng_3^1 is compounded of g_3^1, $|\alpha p + \beta q|$ is also compounded of g_3^1. That is, if $\dim |\alpha p + \beta q| = k$, then $|\alpha p + \beta q| = kg_3^1 + B = k|2p+q| + B$. But $|\alpha p + \beta q|$ is base point free, thus $B = 0$. Hence $(\alpha, \beta) = (2k, k)$ for some $k = 0, 1, \cdots, n$.

THEOREM 3.4. *Let p and q be non-ramification points of g_3^1 and $p+q+u \in g_3^1$ for some $u \in C$. Then there are no elements of $H(p,q)$ in $N_{\leq n} \times N_{\leq n}$ except for $(0,0)$.*

Proof: Notice that $p \neq u$ and $q \neq u$. For each nonnegative integer $k < n$, u is not a base point of the linear series $ng_3^1 - ku = |(n-k)g_3^1 + kp + kq|$, hence

$$\dim |ng_3^1 - (k+1)u| = \dim |ng_3^1| - (k+1) = n - (k+1).$$

Thus

$$\dim |np + nq| = \dim |ng_3^1| - n = 0.$$

Hence $\dim |\alpha p + \beta q| = 0$, for all $\alpha \leq n, \beta \leq n$.

THEOREM 3.5. *Let p and q be points on C such that there is no element in g_3^1 containing $p+q$. Then $\alpha + \beta \geq (g+2)/2$, for each element $(\alpha, \beta) \in H(p,q)$ other than $(0,0)$.*

Proof: Let $(\alpha, \beta) \in H(p,q)$ and $(\alpha, \beta) \neq (0,0)$. Since $|\alpha p + \beta q|$ is not compounded of g_3^1 the residual series is compounded of g_3^1, by Lemma 3.1. Now the result follows from Theorem 3.2.

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