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NONLINEAR ERGODIC THEOREMS OF SEMIGROUPS OF NONEXPANSIVE MAPPINGS

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1. Introduction

Let G be a commutative semigroup. Then, since every two ideals in G have non-void intersection, $(G, \succeq)$ is a directed system when the binary relation "$\succeq$" on G is defined by $t \succeq s$ if and only if $t \in \{s\} \cup Gs$, $s, t \in G$. Let $m(G)$ be the Banach space of all bounded real valued functions on G and let D be a subspace of $m(G)$ containing constants and invariant under r_s, $s \in G$, where $(r_s f)(t) = f(ts)$ for every $f \in m(G)$. A net $\{\mu_\alpha\}$ of continuous linear functionals on D is called *strongly regular* if it satisfies the following conditions:

(a) $\sup_\alpha \|\mu_\alpha\| < +\infty$;

(b) $\lim_\alpha \mu_\alpha(1) = 1$;

(c) $\lim_\alpha \|\mu_\alpha - r_s^* \mu_\alpha\| = 0$ for every $s \in G$.

Let E be a uniformly convex Banach space and let $\Im = \{S(t) : t \in G\}$ be a representation of G as nonexpansive mappings on a closed convex subset C of E into C, i.e., $S(ts)x = S(t)S(s)x$ for all $t, s \in G$ and $x \in C$. Let $F(\Im)$ denote the set of common fixed points of $\Im$ in C, i.e., $\{x \in C : S(t)x = x \text{ for all } t \in G\}$. Then, as well known, $F(\Im)$ (possibly empty) is a closed convex subset of C; see [1;Theorem 8].

As the title suggests, the purpose of this paper is to provide the nonlinear ergodic theorems for commutative semigroups of nonexpansive mappings in Banach spaces. First, we prove the exsistence of a nonexpansive retraction P of C onto $F(\Im)$ such that $PS(t) = S(t)P = P$ for every $t \in G$ and $Px \in \overline{co}\{S(t)x : t \in G\}$ for each $x \in C$, where $\overline{co}A$ denotes the closure of the convex hull of A. Secondly, we show that if

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E has a Fréchet differentiable norm, then, such a retraction P is unique and further if D is a subspace of $m(G)$ containing all the functions h_{x^*} on G with $x \in C$, $x^* \in E^*$ given by $h_{x^*}(t) =< S(t)x, x^* >$ for each $t \in G$, then for any strongly regular net $\{\mu_\alpha\}$ of continuous linear functions on D, $\Im_{\mu_\alpha} x$ converges weakly to Px for each $x \in C$, where $\Im_{\mu_\alpha}$ is a mapping of C into $F(\Im)$ such that $< \Im_{\mu_\alpha} x, x^* >= \mu_\alpha(h_{x^*})$ for every $x \in C$ and $x^* \in E^*$.

2. Preliminaries and Some Lemmas

Throughout this paper, we assume that a Banach space E is real. We denote by E^* the dual space of E and the value of $x^* \in E^*$ at $x \in E$ will be denoted by $< x, x^* >$. Let D be a subset of E. Then, $\overline{co}D$ is the closure of the convex hull of D.

Let G be a commutative semigroup and let $m(G)$ be the Banach space of all bounded real valued functions on G with the supremum norm. Then, for each $s \in G$ and $f \in m(G)$, we can define $r_s f$ in $m(G)$ by $(r_s f)(t) = f(ts)$ for all $t \in G$. Let D be a subspace of $m(G)$ containing constants. A linear functional μ on D is called a *mean* on D if $\|\mu\| = \mu(1) = 1$. Further, if D is invariant under every r_s, $s \in G$, then a mean μ on D is *invariant* if $\mu(r_s f) = \mu(f)$ for all $s \in G$ and $f \in D$. For $s \in G$, we can define a point evaluation δ_s by $\delta_s(f) = f(s)$ for every $f \in m(G)$. A convex combination of point evaluations is called a *finite mean* on G. A finite mean μ on G is also a mean on any subspace D of $m(G)$ containing constants. Then we have the following:

LEMMA 2.1. *Let $f : G \to E$ be a function such that the weak closure of $\{f(t) : t \in G\}$ is weakly compact and let D be a subspace of $m(G)$ containing all the functions h_{x^*} on G with $x^* \in E^*$ given by $h_{x^*}(t) =< f(t), x^* >$ for each $t \in G$. Then, for any $\mu \in D^*$, there exists an element f_μ in E such that*

$$< f_\mu, x^* >= \int < f(t), x^* > d\mu(t)$$

for all $x^* \in E^*$, where $\int < f(t), x^* > d\mu(t) = \mu(h_{x^*})$. (Such an f_μ will be written by $\int f(t) d\mu(t)$).

Proof. We define a functional F on E^* by

$$F(x^*) = \int < f(t), x^* > d\mu(t)$$

for every $x^* \in E^*$. Then, F is an element of E^{**} since

$$\begin{aligned} |F(x^*)| &= \Big| \int < f(t), x^* > d\mu(t) \Big| \\ &\leq \sup_{t \in G} | < f(t), x^* > | \cdot \|\mu\| \\ &\leq (\sup_{t \in G} \|f(t)\|) \|x^*\| \cdot \|\mu\| \end{aligned}$$

for every $x^* \in E^*$. Now to show $F \in E$, let $K_o = \overline{co}\{f(t) : t \in G\}$. Then, K_o is weakly compact. Further let $K_1 = \{\|\mu\| x : x \in K_o\}$, $K_2 = \{ry : 0 \leq r \leq 1, y \in K_1\}$. Finally, let $K_3 = K_2 - K_2$. Then, K_3 is a circled weakly compact convex subset of E. If n is the natural embedding of E into E^{**}, then $n(K_3)$ is also a circled weak* compact convex subset of E^{**}. Thus, it is sufficient to show $F \in n(K_3)$. If not, then, by the separation theorem, there is an $x^* \in E^*$ such that

$$\sup\{| < z^{**}, x^* > | : z^{**} \in n(K_3)\} < F(x^*).$$

On the other hand,

$$\begin{aligned} &\sup\{| < z^{**}, x^* > | : z^{**} \in n(K_3)\} = \sup\{| < z, x^* > | : z \in K_3\} \\ &\geq \sup\{| < z, x^* > | : z \in K_2\} = \sup\{r | < y, x^* > | : 0 \leq r \leq 1, y \in K_1\} \\ &\geq \sup\{| < \|\mu\| f(t), x^* > | : t \in G\} = \|\mu\| \sup_{t \in G} | < f(t), x^* > | \\ &\geq \Big| \int < f(t), x^* > d\mu(t) \Big| = |F(x^*)|, \end{aligned}$$

which gives a contradiction. Hence, for each $\mu \in D^*$, there exists an $f_\mu \in K_3$ such that $< f_\mu, x^* >= F(x^*) = \int < f(t), x^* > d\mu(t)$ for every $x^* \in E^*$. This completes the proof.

We note that μ is a mean on D if and only if

$$(2.1) \qquad \inf_{t\in G} f(t) \le \mu(f) \le \sup_{t\in G} f(t)$$

for every $f \in D$; see [3],[5].

LEMMA 2.2. *Let f, D be given as Lemma 2.1. Furthermore, if D contains constants and if μ is a mean on D, then $f_\mu = \int f(t)d\mu(t) \in \overline{co}\{f(t) : t \in G\}$.*

Proof. Let F, K_o, n be as in the proof of Lemma 2.1. Suppose $F \notin n(K_o)$. Then there is an $x^* \in E^*$ such that

$$\sup\{< z^{**}, x^* >: z^{**} \in n(K_o)\} < F(x^*).$$

But, by (2.1),

$$F(x^*) = \int < f(t), x^* > d\mu(t) \le \sup_{t\in G} < f(t), x^* >$$
$$\le \sup\{< z, x^* >: z \in K_o\} = \sup\{< z^{**}, x^* >: z^{**} \in n(K_o)\},$$

which gives a contradiction. Hence, $f_\mu = \int f(t)d\mu(t) \in K_o$.

The following two lemmas are crucial to prove nonlinear ergodic theorems for commutative semigroups of nonexpansive mappings in a Banach space.

LEMMA 2.3. *Let G be a commutative semigroup and let D be a subspace of $m(G)$ containing constants and invariant under r_s, $s \in G$. Let E be a Banach space and let $f : G \to E$ be a function such that for each $x^* \in E^*$, a function $h_{x^*} : t \mapsto < f(t), x^* >$ is in D and the weak closure of $\{f(t) : t \in G\}$ is weakly compact. Then, for a weakly compact convex set $K \subset E$, the following are equivalent:*

- (a) *for any weak neighborhood W of K, there exists a finite mean λ on G such that $\int f(ts)d\lambda(t) \in W$ for every $s \in G$;*
- (b) *there is a net $\{\lambda_\alpha\}$ of finite means on G such that for any weak neighborhood W of K, there is α_o with $\int f(st)d\lambda_\alpha(t) \in W$ for every $\alpha \succeq \alpha_o$ and $s \in G$;*
- (c) *for any invariant mean μ on $m(G)$, $f_\mu = \int f(t)d\mu(t) \in K$;*
- (d) *for any invariant mean μ on D, $f_\mu = \int f(t)d\mu(t) \in K$.*

Proof. (b)$\Longrightarrow$(a) and (d)$\Longrightarrow$(c) are clear. To prove (a)$\Longrightarrow$(d), let μ be an invariant mean on D and let W be a weak neighborhood of K. Then, it is easy to see that there is a weakly closed convex neighborhood W' of K with $W \supset W'$. For such a W', it follows from (a) that there exists a finite mean $\lambda = \sum_{i=1}^{n} a_i \delta_{t_i}$ on G with $a_1, \cdots, a_n \geq 0$, $\sum_{i=1}^{n} a_i = 1$ and $t_1, \cdots, t_n \in G$ such that $\sum_{i=1}^{n} a_i f(t_i s) \in W'$ for every $s \in G$. Since μ is invariant, we have

$$\begin{aligned} < f_\mu, x^* > &= \int < f(t), x^* > d\mu(t) \\ &= \sum_{i=1}^{n} a_i \int < f(t_i t), x^* > d\mu(t) \\ &= \int < \sum_{i=1}^{n} a_i f(t_i t), x^* > d\mu(t) \\ &= < \int \sum_{i=1}^{n} a_i f(t_i t) d\mu(t), x^* > \end{aligned}$$

for every $x^* \in E$. Then, by Lemma 2.2, we have

$$f_\mu = \int f(t) d\mu(t) = \int \sum_{i=1}^{n} a_i f(t_i t) d\mu(t) \in W'.$$

Since K is weakly closed, we have $f_\mu = \int f(t) d\mu(t) \in K$.

Now to show (c)$\Longrightarrow$(b), By Theorem 1 of Day [3], there exists a net $\{\lambda_\alpha\}$ of finite means on G such that $\|\lambda_\alpha - r_s^* \lambda_\alpha\| \to 0$ for each $s \in G$, where r_s^* is the conjugate operator of r_s. If we deny (b), then there exists a weakly open neighborhood W of K such that, for every α, there is $\beta_\alpha \succeq \alpha$ and $s_\alpha \in G$ with

$$f_{r_{s_\alpha}^* \lambda_{\beta_\alpha}} = \int f(t s_\alpha) d\lambda_{\beta_\alpha}(t) \notin W. \tag{2.2}$$

By taking a subnet of $\{r^*_{s_\alpha}\lambda_{\beta_\alpha}\}$, if necessary, we may assume that $r^*_{s_\alpha}\lambda_{\beta_\alpha}$ converges to $\eta \in m(G)^*$ in the weak* topology. Then η is an invariant mean on $m(G)$. Indeed, for each $s \in G$, we have

$$\|r^*_{s_\alpha}\lambda_{\beta_\alpha} - r^*_s r^*_{s_\alpha}\lambda_{\beta_\alpha}\| = \|r^*_{s_\alpha}(\lambda_{\beta_\alpha} - r^*_s\lambda_{\beta_\alpha})\| \leq \|\lambda_{\beta_\alpha} - r^*_s\lambda_{\beta_\alpha}\| \to 0.$$

So, $f_{r^*_{s_\alpha}\lambda_{\beta_\alpha}}$ converges weakly to $f_\eta \in K$ by (c), which gives a contradiction to (2.2).

Let G be a commutative semigroup and let D be a subspace of $m(G)$ containing constants and invariant under every r_s, $s \in G$. Then, a net $\{\mu_\alpha\}$ of continuous linear functionals on D is called *strongly regular* if it satisfies the following conditions :

(a) $\sup_\alpha \|\mu_\alpha\| < +\infty$;
(b) $\lim_\alpha \mu_\alpha(1) = 1$;
(c) $\lim_\alpha \|\mu_\alpha - r^*_s\mu_\alpha\| = 0$ for every $s \in G$.

LEMMA 2.4. *Let G, D, E and f be as in Lemma 2.3. Let $y \in E$. Then the following are equivalent :*

- (â) *for any weak neighborhood W of y, there exists a finite mean λ on G such that $\int f(ts)d\lambda(t) \in W$ for every $s \in G$;*
- (e) *for a strongly regular net $\{\mu_\alpha\}$ of continuous linear functionals on D, $\int f(ts)d\mu_\alpha(t)$ converges weakly to y uniformly in $s \in G$.*

Proof. (e)$\Longrightarrow$ (â). By [3], there exists a net $\{\lambda_\alpha\}$ of finite means on G such that $\|\lambda_\alpha - r^*_s\lambda_\alpha\| \to 0$ for each $s \in G$. Then, the net $\{\lambda_\alpha\}$ of continuous linear functionals on D is certainly strongly regular. By (e), for any weak neighborhood W of y, there exists α_o such that

$$\int f(st)d\lambda_\alpha(t) \in W \quad \text{for every } \alpha \succeq \alpha_o \text{ and } s \in G.$$

Taking a finite mean λ_{α_o}, we obtain the desired result.

(â)$\Longrightarrow$(e). Fix $x^* \in E^*$ and $\epsilon > 0$ arbitrarily. Then, by (â) there exists a finite mean $\lambda = \sum_{i=1}^n a_i\delta_{t_i}$ on G, where $a_1, \cdots, a_n \geq 0$ with $\sum_{i=1}^n a_i = 1$ and $t_1, \cdots, t_n \in G$, such that

$$| < \sum_{i=1}^n a_i f(t_i s) - y, x^* > | < \frac{\epsilon}{\sup_\alpha \|\mu_\alpha\|} \quad \text{for every } s \in G.$$

Note that, for each $h \in G$,

$$\int\int f(tsh)d\lambda(t)d\mu_\alpha(s) = \int \sum_{i=1}^n a_i f(t_i sh)d\mu_\alpha(s)$$

is well-defined. Then, we have

$$\begin{aligned} &< \int\int f(tsh)d\lambda(t)d\mu_\alpha(s) - y, x^* > \\ &= \int < \int f(tsh)d\lambda(t), x^* > d\mu_\alpha(s) - < y, x^* > \\ &= \int < y, x^* > d\mu_\alpha(s) + \int < \int f(tsh)d\lambda(t) - y, x^* > d\mu_\alpha(s) - < y, x^* > \end{aligned}$$

Since $\{\mu_\alpha\}$ is strongly regular, there exists α_o such that

$$|1 - \mu_\alpha(1)| < \epsilon / \max\{1, \|y\| \cdot \|x^*\|\}$$

and

$$(2.3) \qquad \|\mu_\alpha - r_{t_i}^* \mu_\alpha\| < \epsilon / \max\{1, M \cdot \|x^*\|\}, \;\; i = 1, 2, \cdots, n,$$

where $M = \sup_{s \in G} \|f(s)\|$. Then we have

$$| < y, x^* > - \int < y, x^* > d\mu_\alpha(s)| \leq | < y, x^* > | \cdot |1 - \mu_\alpha(1)| < \epsilon$$

for every $\alpha \succeq \alpha_o$ and

$$|\int < \int f(tsh)d\lambda(t) - y, x^* > d\mu_\alpha(s)|$$
$$\leq \|\mu_\alpha\| \cdot \Big| < \sum_{i=1}^{n} a_i f(t_i sh) - y, x^* > \Big| < \epsilon$$

for every $h \in G$ and α. Thus, we obtain

$$| < \int \int f(tsh)d\lambda(t)d\mu_\alpha(s) - y, x^* > | < 2\epsilon$$

for every $h \in G$ and $\alpha \succeq \alpha_o$. On the other hand, it follows from (2.3) that

$$| < \int f(sh)d\mu_\alpha(s), x^* > - < \int \int f(tsh)d\lambda(t)d\mu_\alpha(s), x^* > |$$
$$= | \int < f(sh) - \sum_{i=1}^{n} a_i f(t_i sh), x^* > d\mu_\alpha(s)|$$
$$\leq \sum_{i=1}^{n} a_i \Big| \int < f(sh) - f(t_i sh), x^* > d\mu_\alpha(s) \Big|$$
$$= \sum_{i=1}^{n} a_i \Big| \int < f(sh), x^* > d(\mu_\alpha - r_{t_i}^* \mu_\alpha)(s) \Big|$$
$$\leq \sum_{i=1}^{n} a_i \|\mu_\alpha - r_{t_i}^* \mu_\alpha\| \cdot M \cdot \|x^*\| < \epsilon$$

for every $h \in G$ and $\alpha \succeq \alpha_o$. Therefore, we have

$$\Big| < \int f(sh)d\mu_\alpha(s) - y, x^* > \Big| < 3\epsilon$$

for every $h \in G$ and $\alpha \succeq \alpha_o$. This completes the proof.

3. Nonlinear Ergodic Theorems

Let G be a commutative semigroup and let C be a closed convex subset of a Banach space E. Then a family $\Im = \{S(t) : t \in G\}$ is called a *representation* of G as nonexpansive mappings on C into itself if $S(st) = S(s)S(t)$ for all $s, t \in G$. Let D be a subspace of $m(G)$ containing constants and invariant under every $r_s, s \in G$. Assume that, for each $x \in C$ and $x^* \in E^*$, a function $t \mapsto < S(t)x, x^* >$ is in D and the weak closure of $\{S(t)x : t \in G\}$ is weakly compact. Then for any $\mu \in D^*$, we can consider a mapping $\Im_\mu$ of C into E such that

$$< \Im_\mu x, x^* >= \int < S(t)x, x^* > d\mu(t)$$

for every $x \in C$ and $x^* \in E^*$; see section 2. Particularly, if μ is a mean on D, then $\Im_\mu$ is a nonexpansive mapping of C into itself. Futhermore, if μ is finite, say $\mu = \sum_{i=1}^{n} a_i \delta_{s_i}$ $(s_i \in G,\ a_i \geq 0,\ i = 1, 2, \cdots, n,\ \sum_{i=1}^{n} a_i = 1)$, then $\Im_\mu x = \sum_{i=1}^{n} a_i S(s_i)x$. Let $F(\Im)$ denote the set of all fixed points of $\Im = \{S(t) : t \in G\}$. Then, we have the following;

THEOREM 3.1. *Let G be a commutative semigroup and let D be a subspace of $m(G)$ containing constants and invariant under every $r_s, s \in G$. Let C be a closed convex subset of a uniformly convex Banach space E and let $\Im = \{S(t) : t \in G\}$ be a representation of G as nonexpansive mappings of C into itself such that a function $t \mapsto < S(t)x, x^* >$ is in D for each $x \in C$ and $x^* \in E^*$ and $F(\Im) \neq \emptyset$. Then, for every invariant mean μ on D, $\Im_\mu$ is a nonexpansive retraction of C onto $F(\Im)$ such that $\Im_\mu S(t) = S(t)\Im_\mu = \Im_\mu$ for each $t \in G$ and $\Im_\mu x \in \overline{co}\{S(t)x : t \in G\}$ for each $x \in C$.*

Proof. Let $x \in C$. Then, we know from [4;Lemma 3] that for each finite mean λ on G.

$$\lim_t \|S(s)\Im_\lambda S(t)x - \Im_\lambda S(st)x\| = 0$$

uniformly in $s \in G$. Let $\{\lambda_\alpha\}$ be a net of finite means on G such that $\|\lambda_\alpha - r_s^*\lambda_\alpha\| \to 0$ for every $s \in G$; see [3]. Then, for $\epsilon > 0$ and $s \in G$, consider α such that $\|\lambda_\alpha - r_s^*\lambda_\alpha\| < \frac{\epsilon}{M}$, where $M = \sup_{t\in G}\|S(t)x\|$. For such an α, there exists t_o such that

$$\|S(s)\Im_{\lambda_\alpha}S(t_o t)x - \Im_{r_s^*\lambda_\alpha}S(t_o t)x\| < \epsilon$$

for every $t \in G$. Hence

$$\begin{aligned}
&\|S(s)\Im_{r_{t_o}^*\lambda_\alpha}S(t)x - \Im_{r_{t_o}^*\lambda_\alpha}S(t)x\| \\
\leq &\|S(s)\Im_{r_{t_o}^*\lambda_\alpha}S(t)x - \Im_{r_s^*\lambda_\alpha}S(t_o t)x\| + \|\Im_{r_s^*\lambda_\alpha}S(t_o t)x - \Im_{r_{t_o}^*\lambda_\alpha}S(t)x\| \\
< &\epsilon + M \cdot \frac{\epsilon}{M} = 2\epsilon
\end{aligned}$$

for every $t \in G$. On the other hand, we know from [2;Lemma 1.4] that for each weak neighborhood W of $\overline{co}\{S(t)x : t \in G\} \cap F(S(s))$, there is $\epsilon > 0$ such that $\|x - S(s)x\| < \epsilon \Longrightarrow x \in W$. Then from the above there is a finite mean λ_α on G such that $\Im_{r_{t_o}^*\lambda_\alpha}S(t)x \in W$ for every $t \in G$. So, for an invariant mean μ on D, by using Lemma 2.3 we have

$$\Im_\mu x \in \overline{co}\{S(t)x : t \in G\} \cap F(S(s)) \subset F(S(s)).$$

Since s is arbitrary, we have $\Im_\mu x \in F(\Im) = \bigcap_{s\in G} F(S(s))$. Thus $\Im_\mu$ is a nonexpansive retraction of C onto $F(\Im)$. From

$$\begin{aligned}
< \Im_\mu S(s)x, x^* > &= \int < S(ts)x, x^* > d\mu(t) \\
&= \int < S(t)x, x^* > d\mu(t) \\
&=< \Im_\mu x, x^* >
\end{aligned}$$

for every $s \in G, x \in C$ and $x^* \in E^*$, we have $\Im_\mu S(s) = \Im_\mu$ for every $s \in G$. Since μ is a mean on D, by Lemma 2.2, $\Im_\mu x$ is contained in $\overline{co}\{S(t)x : t \in G\}$ for each $x \in C$. This completes the proof.

THEOREM 3.2. *Let G, D, C, E and $\Im = \{S(t) : t \in G\}$ be as in Theorem 3.1. Additionally, assume that E has a Frećhet differentiable norm. Then there is a unique nonexpansive retraction P of C onto $F(\Im)$ such that $PS(t) = S(t)P = P$ for each $t \in G$ and $Px \in \overline{co}\{S(t)x : t \in G\}$*

for each $x \in C$. Further, if $\{\mu_\alpha\}$ is a strongly regular net of continuous linear functionals on D, then for each $x \in C$, $\Im_{\mu_\alpha} S(t)x$ converges weakly to Px uniformly in $t \in G$.

Proof. By Theorem 3.1, there is a nonexpansive retraction P of C onto $F(\Im)$ such that $PS(t) = S(t)P = P$ for each $t \in G$ and $Px \in \overline{co}\{S(t)x : t \in G\}$ for each $x \in C$. Fix $x \in C$ and $s \in G$. Then we have

$$\begin{aligned} Px = PS(s)x &\in \overline{co}\{S(ts)x : t \in G\} \\ &= \overline{co}\{S(t)x : t \succeq s,\ t \in G\}. \end{aligned}$$

Hence, $Px \in \bigcap_{s \in G} \overline{co}\{S(t)x : t \succeq s\}$. By [6;Theorem 1], we also know $\bigcap_{s \in G} \overline{co}\{S(t)x : t \succeq s\} \cap F(\Im)$ consists of at most one point. Therefore we know

$$\{Px\} = \bigcap_{s \in G} \overline{co}\{S(t)x : t \succeq s\} \cap F(\Im)$$

for every $x \in C$. This implies that such a retraction P is unique. Let $x \in C$. Then, for any invariant mean μ on D, it follows from Theorem 3.1 and the above that

$$\int S(s)x \; d\mu(s) = \Im_\mu x = Px.$$

So, (d) of Lemma 2.3 is satisfied with $K = \{Px\}$. Therefore, by (e) of Lemma 2.4, for any strongly regular net $\{\mu_\alpha\}$ of continuous linear functionals on D,

$$\int S(s)S(t)x \; d\mu_\alpha(s) = \Im_{\mu_\alpha} S(t)x$$

converges weakly to Px uniformly in $t \in G$, which completes the proof.

4. Some Applications.

In this section, by using Theorem 3.2, we prove some nonlinear ergodic theorems for nonexpansive mappings and nonexpansive semigroups in

Banach spaces. In what follows let C, E be as in Theorem 3.2. We start with the following ;

THEOREM 4.1. *Let T be a nonexpansive mapping of C into itself with $F(T) \neq \phi$. Then, for each $x \in C$, $\frac{1}{n}\sum_{i=0}^{n-1} T^{i+k}x$ converges weakly to some $y \in F(T)$, as $n \to \infty$, uniformly in $k = 0, 1, 2, \cdots$.*

Proof. Let $G = \{0, 1, 2, \cdots\}, \Im = \{T^i : i \in G\}, D = m(G)$, and $\mu_n(f) = \frac{1}{n}\sum_{i=0}^{n-1} f(i)$ for $n = 1, 2, \cdots$ and $f \in D$. Then, since

$$\begin{aligned} \|\mu_n - r_1^*\mu_n\| &= \sup_{\|f\|\leq 1} |(\mu_n - r_1^*\mu_n)(f)| \\ &= \frac{1}{n}\sup_{\|f\|\leq 1} |f(o) - f(n)| \leq \frac{2}{n} \to 0, \end{aligned}$$

as $n \to \infty$, we obtain Theorem 4.1 by using Theorem 3.2.

Let $\Im = \{S(t) : 0 \leq t < +\infty\}$ be a family of nonexpansive mappings of C into itself such that $S(0) = I, S(t+s) = S(t)S(s)$ for all $t, s \in [0, \infty)$ and $S(t)x$ is continuous in $t \in [0, \infty)$ for each $x \in C$. Then, $\Im = \{S(t) : 0 \leq t < +\infty\}$ is said to be a *nonexpansive semigroup* on C. Then, as a direct consequence of Theorem 3.2, we have the following ;

THEOREM 4.2. *Let $\Im = \{S(t) : 0 \leq t < +\infty\}$ be a nonexpansive semigroup on C with $F(\Im) \neq \emptyset$. Then for each $x \in C$, $\frac{1}{s}\int_o^s S(t+k)x\, dt$ converges weakly to some $y \in F(\Im)$, as $n \to \infty$, uniformly in $k \geq 0$.*

Proof. Let $G = [0, \infty), \Im = \{S(t); 0 \leq t < \infty\}$, and let D be the Banach space $C(G)$ of bounded continuous functions on G. Define $\mu_s(f) = \frac{1}{s}\int_o^s f(t)\, dt$ for every $s > 0$ and $f \in D$. Then we obtain that

$$\begin{aligned}\|\mu_s - r_k^*\mu_s\| &= \sup_{\|f\|\le 1}\left|\frac{1}{s}\int_o^s f(t)dt - \frac{1}{s}\int_o^s f(t+k)dt\right| \\ &= \frac{1}{s}\sup_{\|f\|\le 1}\left|\int_o^s f(t)dt - \int_k^{s+k} f(t)dt\right| \\ &= \frac{1}{s}\sup_{\|f\|\le 1}\left|\int_o^k f(t)dt - \int_s^{s+k} f(t)dt\right| \\ &\le \frac{1}{s}\sup_{\|f\|\le 1}\left(\int_o^k |f(t)|dt + \int_s^{s+k} |f(t)|dt\right| \\ &= \frac{2k}{s} \to 0,\end{aligned}$$

as $s \to \infty$. Therefore by using Theorem 3.2, we have Theorem 4.2.

Let $G = \{0, 1, 2, \cdot\}$ and let $Q = \{q_{n,m}\}_{n,m\in N}$ be a matrix satisfying the following conditions:

(a) $\sup_{n\ge 0}\sum_{m=o}^{\infty} |q_{n,m}| < +\infty$;

(b) $\lim_{n\to\infty}\sum_{m=o}^{\infty} q_{n,m} = 1$;

(c) $\lim_{n\to\infty}\sum_{m=o}^{\infty} |q_{n,m+1} - q_{n,m}| = 0.$

Then, Q is called a strongly regular matrix ; see [7]. If Q is a strongly regular matrix, then for each $m \in G$, we have $|q_{n,m}| \to 0$, as $n \to \infty$. Indeed, assume that there is $m_o \in G$ such that $|q_{n,m_o}| \not\to 0$, as $n \to \infty$. Then there are $\epsilon > 0$ and a subsequence $\{|q_{n_i,m_o}|\}$ of $\{|q_{n,m_o}|\}$ with $|q_{n_i,m_o}| > \epsilon$. On the other hand, since Q is a strongly regular matrix, there exists $n_o \in G$ such that

$$\sum_{m=o}^{\infty} |q_{n,m+1} - q_{n,m}| < \frac{\epsilon}{2}$$

for every $n \ge n_o$. So, we have $|q_{n,\ell} - q_{n,m}| < \frac{\epsilon}{2}$ for every $n \ge n_o$ and $l, m \in G$. Fix n_i with $n_i \ge n_o$. Then we have

$$|q_{n_i,m}| \geq |q_{n_i,m_o}| - |q_{n_i,m_o} - q_{n_i,m}| > \frac{\epsilon}{2}$$

for every $m \in N$. Therefore we obtain $\sum_{m=o}^{\infty} |q_{n_i,m}| = \infty$, which is a contradiction to (a).

THEOREM 4.3.. *Let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. If Q is a strongly regular matrix, then for each $x \in C$, $\sum_{m=o}^{\infty} q_{n,m} T^{m+k} x$ converges weakly to some $y \in F(T)$, as $n \to \infty$, uniformly in $k = 0, 1, 2, \cdots$.*

Proof. Let $G = \{0, 1, 2, \cdots\}$, $\Im = \{T^n : n \in G\}$, $D = m(G)$, and $\mu_n(f) = \sum_{m=o}^{\infty} q_{n,m} f(m)$ for each $n = 1, 2, \cdots$ and $f \in m(G)$. Then, since Q is a strongly regular matrix, we have that

$$\begin{aligned}
\sup_{n \geq o} \|\mu_n\| &= \sup_{n \geq o} \sup_{\|f\| \leq 1} |\mu_n(f)| \\
&\leq \sup_{n \geq o} \sup_{\|f\| \leq 1} \Big(\sum_{m=o}^{\infty} |q_{n,m}| \cdot |f(m)| \Big) \\
&\leq \sup_{n \geq o} \sum_{m=o}^{\infty} |q_{n,m}| < +\infty
\end{aligned}$$

and by (b),

$$\lim_{n \to \infty} \mu_n(1) = \lim_{n \to \infty} \sum_{m=o}^{\infty} q_{n,m} = 1.$$

We also have $\|\mu_n - r_k^* \mu_n\| \to 0$ for every $k = 0, 1, 2, \cdots$. Indeed, we have $\lim_{n \to \infty} \|r_k^* \mu_n - r_{k+1}^* \mu_n\| \to 0$ for every $k = 0, 1, 2, \cdots$, since

$$\begin{aligned}\|r_k^*\mu_n - r_{k+1}^*\mu_n\| &= \sup_{\|f\|\le 1} |(r_k^*\mu_n - r_{k+1}^*\mu_n)(f)| \\ &= \sup_{\|f\|\le 1}\Big|\sum_{m=o}^{\infty} q_{n,m}\{f(m+k) - f(m+k+1)\}\Big| \\ &= \sup_{\|f\|\le 1}\Big|q_{n,o}f(k) + \sum_{m=o}^{\infty}(q_{n,m+1} - q_{n,m})f(m+k+1)\Big| \\ &\le \sum_{m=o}^{\infty} |q_{n,m+1} - q_{n,m}| + |q_{n,o}|.\end{aligned}$$

Further for $k \ge 2$, we have

$$\|\mu_n - r_k^*\mu_n\| \le \sum_{i=1}^{k} \|r_i^*\mu_n - r_{i-1}^*\mu_n\| \;\rightarrow 0,$$

as $n \to \infty$. Therefore by applying Theorem 3.2, we have Theorem 4.3.

THEOREM 4.4.. *Let S and T be nonexpansive mappings of C into itself with $ST = TS$ and $F(T) \cap F(S) \ne \emptyset$. Then, for each $x \in C$, $\frac{1}{n^2}\sum_{i,j=o}^{n-1} S^{i+j}T^{j+h}x$ converges weakly to some $y \in F(T) \cap F(S)$,* as $n \to \infty$, *uniformly in $k, h = 0, 1, 2, \cdots$.*

Proof. Let $G = \{0,1,2,\cdots\} \times \{0,1,2,\cdots\}, \Im = \{S^iT^j : (i,j) \in G\}, D = m(G)$ and $\mu_n(f) = \frac{1}{n^2}\sum_{i,j=o}^{n-1} f(i,j)$ for each $n = 1, 2, \cdots$ and $f \in m(G)$. Then, for each $(\ell, m) \in G$, we have

$$\begin{aligned}\|\mu_n - r_{(\ell,m)}^*\mu_n\| &= \sup_{\|f\|\le 1} |(\mu_n - r_{(\ell,m)}^*\mu_n)(f)| \\ &= \sup_{\|f\|\le 1}\Big|\frac{1}{n^2}\sum_{i,j=o}^{n-1} f(i,j) - \frac{1}{n^2}\sum_{i,j=o}^{n-1} f(i+\ell, j+n)\Big| \\ &\le \frac{1}{n^2}\{\ell \times n + m(n-\ell) + \ell \times n + m(n-\ell)\} \\ &= \frac{1}{n^2}\{2n(\ell+m) - 2m\ell\} \rightarrow 0,\end{aligned}$$

as $n \to \infty$. Therefore by using Theorem 3.2, we have Theorem 4.4.

THEOREM 4.5. *Let* $\Im = \{S(t) : t \geq 0\}$ *be a nonexpansive semigroup on* C *with* $F(\Im) \neq \emptyset$. *Then, for each* $x \in C$, $\lambda \int_o^\infty e^{-\lambda t} S(t+k)x\, dt$ *converges weakly to* $y \in F(\Im)$, *as* $\lambda \downarrow 0$, *uniformly in* $k \geq 0$.

Proof. Let $G = [0, \infty), \Im = \{S(t) : t \geq 0\}, D = C(G)$ and $\mu_\lambda(f) = \lambda \int_o^\infty e^{-\lambda t} f(t) dt$ for each $\lambda > 0$ and $f \in C(G)$.
Then, for each $s \in [0, \infty)$, we have

$$\begin{aligned} \|\mu_\lambda - r_s^* \mu_\lambda\| &= \sup_{\|f\| \leq 1} \left|\lambda \int_o^\infty e^{-\lambda t} f(t) dt - \lambda \int_o^\infty e^{-\lambda t} f(s+t) dt\right| \\ &= \sup_{\|f\| \leq 1} \left|\lambda \int_o^s e^{-\lambda t} f(t) dt + \lambda(1 - e^{\lambda s}) \int_s^\infty e^{-\lambda t} f(t) dt\right| \\ &\leq \lambda s + |1 - e^{-\lambda s}| \to 0, \end{aligned}$$

as $\lambda \downarrow 0$. Therefore by using Theorem 3.2, we have Theorem 4.5.

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