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M–HOMOTOPY EXTENSION PROPERTY AND M–HOMOTOPY GROUPS

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1. Introduction

In [4], R. Jerrad introduced m-functions as generalization of continuous function between topological spaces. m-functions are weighted, finitely valued functions with a property corresponding that of usual continuity. In [6], R. Jerrad and M.D.Meyerson defined the m-homotopy and m-homotopy groups and showed that the n-th m-homotopy group has a natural definition as $m(\pi_n(Y)) = \text{hom }(S^n, Y)$ in a cetrain category of m-functions, which is an R-module under the addition of m-functions for a ring R with identity without zero divisors. They also showed that m-homotopy theory is a homology theory by proving it satisfies the Eilenberg-steenrod axioms.

In this paper we generalize the pasting lemma on continuous functions to that on m-functions and show that for a triangulable pair (X, A), A has certain extension property in X for m-functions similar to absolute homotopy extension property. By using those facts, we prove that the m-fundamental group acts on n-th m-homotopy group as a group automorphism for $n \geq 1$.

2. m-function and m-homotopy group

We introduce some definitions and m-homotopy groups in [4], [5]. Suppose that $\bar{f} : X \times Y \to R$ is a (standard) function, where X and Y are T_2-spaces and R is a ring with identity and without zero divisors. Then we define a multiple-valued function over the ring R $f' : X \to Y$ by its graph $f' = \text{cl}\{(x, y) | \bar{f}(x, y) \neq 0\}$, which satisfies the following condition ;

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(1) for all $x \in X, f'(x) = \{y \in Y | (x,y) \in f'\}$ is a finite or empty subset of Y.

(2) if $f'(x') = \{y_1, \cdots, y_n\}$, there exist disjoint neighborhoods $\hat{V}_i(y_i)$ such that for any neighborhood $V_i(y_i) \subset \hat{V}_i$ there is a neighborhood $U(x')$ satisfying ;

  (a) $\sum_{y \in V_i} \bar{f}(x,y) = \bar{f}(x', y_i)$for $x \in U, i = 1, \cdots, n$

  (b) $\bar{f}(x,y) = 0$ for $x \in U$ and $y \in [Y - \bigcup_{i=1}^{n} V_i]$

(3) if $f'(x') = \phi$, there exists a neighborhood $U(x')$ such that $\bar{f}(x,y) = 0$ for all $x \in U,\ y \in Y$

DEFINITION 2.1. Under the condition above, we define the multiple valued function $f : X \to Y \times R$ given by

$$f = \{(x,(y,r))y, \in f'(x) \text{ and } \bar{f}(x,y) = r\}$$

At this time, f is called an m-function from X to Y defined by the defining function $\bar{f}$(the ring R is usually fixed and dropped from the notation) and R is called weighting factor of f determined by the defining function $\bar{f}$.

The multiplicity of f is $m(f) = \sum_{y \in Y} \bar{f}(x,y)$; it is indendpent of x if X is connected. The empty function, denoted by ϕ is defined by $\bar{\phi} : X \times Y \to 0$. Any continuous function can be regarded as an m-functions by assigning it multipicity one.

The composition of two m-functions $f : X \to Y$ and $g : Y \to Z$ is defined by $\overline{g \circ f}(x,y) = \sum_{y \in Y} \bar{f}(x,y)\bar{g}(x,y)$, so Hausdorff spaces and m-functions over R form a categroy $R - T_2$, with T_2 as a subcategory. Any two m-functions may be added : $f+g$ defined $\overline{f+g} = \bar{f} + \bar{g}$. Also, if $a \in R$ we define the $a \cdot f$ by $\overline{a \cdot f} = a \cdot \bar{f}$. Then $\hom(X,Y)$ is an R-module and there are functors $\hom(-,Z)$ and $\hom(Z,-) : R - T_2 \to (R-\text{modules})$. The restriction of $f : X \to Y$ to a subset $A \subset X$ is defined by $f|_A = f \circ i$ when i is the inclusion $i : A \to X$. An m-function on pairs of Hausdorff spaces is defined as follows : $h : (X,A) \to (Y,B)$ is an m-function on pairs if $h'(A) \subset B$. We say that f is m-homotopic to g relative to $X'(f \sim_m g \text{ rel } X')$ if there exists an m-function $F : (X, A) \times I \to (Y, B)$ with $F(x, 0) = f(x)$, $F(x, 1) = g(x)$ and $F|_{X' \times \{t\}} = f|_{X'} = g|_{X'}$ for $t \in [0, 1]$. An m-function on pointed pairs $f : (X, A, x_0) \to (Y, B, y_0)$ must satisfiy $f|_A : A \to B$ and $f|_{x_0} : x_0 \to y_0$.

LEMMA 2.2. *In the catgory $R_0 - phT_2$ of pointed pairs of Hausdorff spaces and m-homotopy classes of m-functions over R of multiplicity zero, the condition for an m-function to be pointed is equivalent to $f|_{x_0} = \phi$: also, for $f : X \to Y$, $f : (X, A, x_0) \to (Y, y_0, y_0)$ if and only if $f|_A = \phi$. [6].*

For any pair $(X, A) = (X, A, \phi)$ and integer $n \geq 1$, the n-th m-homotopy group $m\pi_n(X, A)$ is defined to have as underlying set, the set of m-homotopy classes of m-functions(of multiplicity zero) $f : (I^n, \partial I^n, 0) \to (X, A)$ where I^n is n-cube, ∂I^n is its boundary and 0 is $\{(0, \cdots, 0)\}$. For $n \geq 1$ the product of f and g in $R_0 - phT_2$, $f \cdot g : I^n \to X$ is defined by

$$\bar{f} \cdot g(b, t, x) = \begin{cases} \bar{f}(b, 2t + 1, x), & 0 \leq t \leq \frac{1}{2} \\ \bar{g}(b, 2t - 1, x) & \frac{1}{2} \leq t \leq 1, \end{cases}$$

where $(b, t, x) \in I^{n-1} \times I \times X$. Especially, $m\pi_1(X)$ is called the m-fundamental group of X.

THEOREM 2.3. *$fg \sim_m f + g$ (where f and g represent elements of $m\pi_n(X, A)$ for $n \geq 1$) [6].*

THEOREM 2.4. *For $n \geq 1$, the homotopy group $m\pi_n(X, A)$ is the R-module $hom((I^n, \partial I^n, 0), (X, A))$, Litting $A = \phi$, $m\pi_n(X) = hom[(I^n, \partial I^n, 0), (X)]$ [6].*

3. A generalization of the pasting lemma

In this section, we prove the following theorem as a generalization of the pasting lemma on continuous functions. Here we let X be a Hausdorff space and R a ring with identity without zero divisors.

THEOREM 3.1. *Let X_1 and X_2 be closed subspaces of X such that $X = X_1 \cup X_2$, $f : X_1 \to Y$ and $g : X_2 \to Y$ m-functions with their* *defining functions* $\bar{f}$, $\bar{g}$ *respectively such that* $\bar{f}(x,y) = \bar{g}(x,y)$ *for every* $x \in X_1 \cap X_2$. *If* $\bar{h} : X \times Y \to R$ *is defined by*

$$\bar{h}(x,y) = \begin{cases} \bar{f}(x,y), & \text{for } (x,y) \in X_1 \times Y, \\ \bar{g}(x,y), & \text{for } (x,y) \in X_2 \times Y, \end{cases}$$

then $h : X \to Y$ *defined by* $\bar{h}$ *is an* m*-function.*

All we have to do to prove the Theorem 3.1 is to show that $\bar{h}$ satisfies the conditions (1), (2) and (3) mentioned in the Definition 2.1. Let's show those facts at following lemmas

LEMMA 3.2. *Under the same hypothesis as the Theorem 3.1,* $h'(x)$ *is finite, for all* $x \in X$.

Proof. Since $f'(x)$ and $g'(x)$ are finite and if $h' = f' \cup g'$, $h'(x) = f'(x) \cup g'(x)$, it is sufficient to show that $h' = f' \cup g'$.

$$\begin{aligned} h' &= \mathrm{cl}\{(x,y) | \bar{h}(x,y) \neq 0\} \\ &= \mathrm{cl}\{(x,y) | \bar{h}(x,y) \neq 0\} \cap (X_1 \times Y \cup X_2 \times Y) \\ &= [(\{(x,y) | \bar{h}(x,y) \neq 0\} \cap X_1 \times Y) \cup (\{(x,y) | \bar{h}(x,y) \neq 0\} \cap X_2 \times Y)] \\ &= [(\{(x,y) | \bar{f}(x,y) \neq 0\} \cap X_1 \times Y) \cup (\{(x,y) | \bar{g}(x,y) \neq 0\} \cap X_2 \times Y)] \\ &\subset (\mathrm{cl}\{(x,y) | \bar{f}(x,y) \neq 0\} \cap X_1 \times Y) \cup (\mathrm{cl}\{(x,y) | \bar{g}(x,y) \neq 0\} \cap X_2 \times Y) \\ &= \mathrm{cl}_{X_1 \times Y}\{(x,y) | \bar{f}(x,y) \neq 0\} \cup \mathrm{cl}_{X_2 \times Y}\{(x,y) | \bar{g}(x,y) \neq 0\} \\ &= f' \cup g' \end{aligned}$$

On the other hand,

$$\begin{aligned} f' &= \mathrm{cl}_{X_1 \times Y}\{(x,y) \in X_1 \times Y | \bar{f}(x,y) \neq 0\} \\ &= \mathrm{cl}\{(x,y) \in X_1 \times Y | \bar{f}(x,y) \neq 0\} \cap X_1 \times Y \\ &= \mathrm{cl}\{(x,y) \in X_1 \times Y | \bar{h}(x,y) \neq 0\} \cap X_1 \times Y \\ &\subset \mathrm{cl}\{(x,y) \in X \times Y | \bar{h}(x,y) \neq 0\} \cap X_1 \times Y \\ &= h' \cap X_1 \times Y \subset h' \end{aligned}$$

Similarly, $g' \subset h'$. Thus $f' \cup g' \subset h'$. Consequently, $h' = f' \cup g'$.

LEMMA 3.3. *Under the same condition as the Theorem 3.1, if* $h'(x') = \{y_1, y_2, \cdots, y_n\}$, *then there exist disjoint neighborhoods* $\hat{V}_i(y_i)$ *such that for any neighborhood* $V_i(y_i) \subset \hat{V}_i$ *there is a neighborhood* $U(x')$ *satisfying ;*

(a) $\sum_{y \in V_i} \bar{h}(x, y) = \bar{h}(x', y_i)$ *for* $x \in U,\ i = 1, 2, \cdots, n.$

(b) $\bar{h}(x, y) = 0$ *for* $x \in U$ *and* $y \in [Y - \bigcup_{i=1}^{n} V_i]$.

Proof. In order to prove the lemma, we consider three cases ; (1) $x' \in X_1 - X_2$ (2) $x' \in X_2 - X_1$ (3) $x' \in X_1 \cap X_2$

The case (1) ; $x' \in X_1 - X_2$. Then $g'(x') = \phi$, $h'(x') = f'(x') = \{y_1, y_2, \cdots, y_n\}$. But f is an m-function with defining function $\bar{f}$. Thus by definition, there exist disjoint neighborhoods $\hat{V}_i(y_i)$ such that for any neighborhoods $V_i(y_i) \subset \hat{V}_i(y_i)$, there is a neighborhood $U'(x')$ in X_1 satisfying ;

(a) $\sum_{y \in V_i} \bar{f}(x, y) = \bar{f}(x', y_i)$ for $x \in U',\ i = 1, 2, \cdots, n.$

(b) $\bar{f}(x, y) = 0$ for $x' \in U'$ and $y \in [Y - \bigcup_{i=1}^{n} V_i]$

Since U' is open in X_1, there is an open $U''(x')$ in X such that $U' = U'' \cap X_1$. Let $U = U'' - X_2$. Then U is open in X and contains x'. But $U \subset U' \subset X_1$. Thus $\sum_{y \in V_i} \bar{h}(x, y) = \sum_{y \in V_i} \bar{f}(x, y) = \bar{f}(x', y_i) = \bar{h}(x', y_i)$ for $x \in U$ and $\bar{h}(x, y) = \bar{f}(x, y) = 0$ for $x \in U \subset U'$ and $y \in (Y - \bigcup_{i=1}^{n} V_i)$.

The proof of the case (2) is similar to that of the case (1).

The case (3) ; Let $x' \in X_1 \cap X_2$ and we note $h'(x') = f'(x') \cup g'(x')$. First, assume $f'(x')$ or $g'(x')$ is empty. Without loss of generality we may assume $f'(x') = \phi$. So we can let $h'(x') = g'(x') = \{y_1, \cdots, y_n\}$. Since g is an m-function, there exist neighborhoods $\hat{V}_i(y_i)$ such that for every neighborhoods $V_i(y_i) \subset \hat{V}_i(y_i)$ there is a neighborhood $U'_{X_2}(x')$ of

x' which is open in X_2 satisfying ;

$$\sum_{y\in V_i} \bar{g}(x,y) = \bar{g}(x',y_i) \text{ for } x\in U'(x'),\ i=1,2,\cdots,n.$$

and

$$\bar{g}(x,y) = 0 \text{ for } x'\in U'(x'),\ y\in [Y-\bigcup_{i=1}^{n} V_i].$$

But since $f'(x')=\phi$, there exist a neighborhood $U''_{X_1}(x')$ of x' which is open in X_1 such that $\bar{f}(x,y)=0$ for $x\in U''_{X_1}(x')$. Let $U(x')=U'(x')\cap U''(x')$, where U' and U'' are open in X such that $U'_{X_1}=U'\cap X_1$ and $U''_{X_2}=U''\cap X_2$ respectively. Then $\sum_{y\in V_i}\bar{h}(x,y)=\sum_{y\in V_i}\bar{g}(x,y)=\bar{g}(x',y_i)=\bar{h}(x',y_i)$ for $x\in U\cap X_2$ and $\sum_{y\in V_i}\bar{h}(x,y)=\sum_{y\in V_i}\bar{f}(x,y)=\bar{f}(x',y_i)=\bar{h}(x',y_i)$ for $x\in U\cap X_1$. Furthermore, if for $x\in U$ and $y\in[Y-\bigcup_{i=1}^{n}V_i]$,

$$\bar{h}(x,y)=\begin{cases}\bar{g}(x,y)=0 & \text{for } x\in U\cap X_2\\ \bar{f}(x,y)=0 & \text{for } x\in U\cap X_1\end{cases}$$

On the other hand, let's assume that $f'(x')$ and $g'(x')$ are not empty. Let $f'(x')=\{y_1^f,\cdots,y_n^f\}$ and $g'(x')=\{y_1^g,\cdots,y_m^g\}$ for $n,\ m\geq 1$. Since f(and g) is a m-function, there are disjoint neighborhoods $\hat{V}_i(y_i^f)$(and $\hat{V}_i(y_i^g)$) such that for any neighborhoods $V_i(y_i^f)\subset\hat{V}_i(y_i^f)$(and $V_i(y_i^g)\subset\hat{V}_i(y_i^g)$) there is a neighborhood $U'_{X_1}(x')$ in X_1(and $U'_{X_2}(x')$ in X_2) satisfying ;

$$\text{(a)} \sum_{y\in V_i} \bar{f}(x,y) = \bar{f}(x',y_i^f) \text{ for } x \in U'_{X_1},\ i = 1,\cdots,n$$

$$\text{(and } \sum_{y\in V_i} \bar{g}(x,y) = \bar{g}(x',y_i^g) \text{ for } x \in U'_{X_2},\ i = 1,\cdots,m)$$

$$\text{(b)}\bar{f}(x,y) = 0 \text{ for } x' \in U'_{X_1} \text{ and } y \in [Y - \bigcup_{i=1}^{n} V_i]$$

$$\text{((and } \bar{g}(x,y) = 0 \text{ for } x' \in U'_{X_2} \text{ and } y \in [Y - \bigcup_{i=1}^{m} V_i]).$$

Without loss of generality, we can take

$$y_1^f = y_2^g, \cdots, y_k^f = y_k^g,\ 0 \leq k \leq \min\{m,n\},$$

where we mean $f'(x') \cap g'(x') = \phi$ by $k = 0$. Let's take $\hat{V}_i = \hat{V}_i(y_i^f) \cap \hat{V}_i(y_i^g)$, $\hat{V}_{k+j} = \hat{V}_{k+j}(y_i^f)$ and $\hat{V}_{k+l} =$ $hatV_{k+l}(y_i^g)$ for $i = 1,\cdots,k,\ j = 1,\cdots,n-k,\ l = 1,\cdots,m-k$. Let

$$y_i = \begin{cases} y_i^f = y_i^g, & \text{for } i = 1,\cdots,k \\ y_{k+j}^f, & \text{for } i = k+1,\cdots,n,\ j = 1,\cdots,n-k \\ y_{k+l}^g, & \text{for } i = n+1,\cdots,n+m-k,\ l = 1,\cdots,m-k \end{cases}$$

Then $h'(x') = f'(x') \cup g'(x') = \{y_1,\cdots,y_{m+n-k}\}$ and $\hat{V}_j$'s are disjoint. If $V_j(y_i) \subset \hat{V}_j(y_j)$, there exists a neighborhood $U'_{X_1}(x')$ (open in X_1) and $U'_{X_2}(x')$ (open in X_2) satisfying ; $\sum_{y\in V_j} \bar{f}(x,y) = \bar{f}(x',y_j)$ for $x \in U'_{X_1}$, $i = 1,\cdots,n$ and $\bar{f}(x,y) = 0$ for $x \in U'_{X_1}$, $y \in [Y - \bigcup_{j=1}^{n} V_j]$. $\sum_{y\in V_j} \bar{g}(x,y) = \bar{g}(x',y_j)$ for $x \in U'_{X_2}$, $j = 1,\cdots,k,\ n+1,\cdots,n+m+k$ and $\bar{g}(x,y) = 0$ for $x \in U'_{X_2}$ and $y \in Y - [(\bigcup_{j=1}^{k} V_j) \cup (\bigcup_{j=n+1}^{n+m-k} V_j)]$. Take

$U = U_{X_1} \cap U_{X_2}$, where U_{X_1} and U_{X_2} are open in X such that $U_{X_1} \cap X_1 = U_{X_1}$, $U'_{X_2} \cap X_2 = U'_{X_2}$ respectively. Then if $l \leq i \leq k$,

$$\sum_{y \in V_i} \bar{h}(x,y) = \sum_{y \in V_i} \bar{f}(x,y) = \bar{f}(x',y_i) = \bar{h}(x',y_i) \text{ for } x \in U \cap X_1,$$
$$\sum_{y \in V_i} \bar{h}(x,y) = \sum_{y \in V_i} \bar{g}(x,y) = \bar{g}(x',y_i) = \bar{h}(x',y_i) \text{ for } x \in U \cap X_2$$

and if $k+1 \leq i \leq n$,

$$\sum_{y \in V_i} \bar{h}(x,y) = \sum_{y \in V_i} \bar{g}(x,y) = 0 = \bar{g}(x',y_i) = \bar{h}(x',y_i) \text{ for } x \in U \cap X_2,$$
$$\sum_{y \in V_i} \bar{h}(x,y) = \sum_{y \in V_i} \bar{g}(x,y) = 0 = \bar{g}(x',y_i) = \bar{h}(x',y_i) \text{ for } x \in U \cap X_2,$$

because $y \in Y - [(\bigcup_{j=1}^{k} V_j) \cup (\bigcup_{j=n+1}^{n+m-k} V_j)]$. Similarly, for $i = n+1, \cdots, m+n-k$, $\sum_{y \in V_i} \bar{h}(x,y) = \bar{h}(x',y_i))$. If $x \in U$ and $y \in [Y - \bigcup_{i=1}^{n+m-k} V_i]$, $\bar{h}(x,y) = 0$. In fact, since $y \in [Y - \bigcup_{i=1}^{n+m-k} V_i] \subset [Y - \bigcup_{i=1}^{n} V_i]$, $\bar{h}(x,y) = \bar{f}(x,y) = 0$ for $x \in U \cap X_1$ and since $y \in [Y - \bigcup_{i=1}^{n+m-k} V_i], \subset [Y - (\bigcup_{i=1}^{k} V_i \cup \bigcup_{i=n+1}^{n+m-k} V_i)]$, $\bar{h}(x,y) = \bar{g}(x,y) = 0$ for $x \in U \cap X_2$.

LEMMA 3.4. *Under the same hypothesis as the Theorem 3.1, if $h'(x') = \phi$ then there exists a neighborhood $U(x')$ such that $\bar{h}(x,y) = 0$ for all $x \in U$, $y \in Y$.*

Proof. Assume $x' \in X_1 - X_2$. Then $h'(x') = f'(x') = \phi$. So there exists a neighborhood $U'_{X_1}(x')$ in X_1 such that $\bar{f}(x,y) = 0$ for all $x \in U_{X_1}$, $y \in Y$. Take $U = U_{X_1} - X_2$, where U_{X_1} is open in X such that $U'_{X_1} = U_{X_1} \cap X_1$. Then $\bar{h}(x,y) = \bar{f}(x,y) = 0$ for $x \in U$, $y \in Y$.

We can prove in case $x' \in X_2 - X_1$ similarly. Assume $x' \in X_1 \cap X_2$. Then since $h'(x') \supset f'(x')$ and $h'(x') \supset g'(x')$ and $h'(x') = \phi$ by hypothesis, $f'(x') = \phi$ and $g'(x') = \phi$. So there exist U'_{X_1} and U'_{X_2} such that $\bar{f}(x,y) = 0$ for $x \in U'_{X_1}$, $y \in Y$ and $\bar{g}(x,y) = 0$ for $x \in U'_{X_2}$ and $y \in Y$. Let $U = U_{X_1} \cap U_{X_2}$, where U_{X_1} and U_{X_2} are open in X such that $U'_{X_1} = U_{X_1} \cap X_1$ and $U'_{X_2} = U_{X_2} \cap X_2$ respectively. Then $\bar{h}(x,y) = 0$ for $x \in U$ and $y \in Y$.

After all, we have completed the proof of Theorem 3.1 by Lemma 3.2, Lemma 3.3 and Lemma 3.4.

COROLLARY 3.5. *Under the same hypothesis as the Theorem 3.1 if f and g are continuous function, then m-function h is continuous*

Proof. We recall that any continuous function f from X to Y has the defining function $\bar{f} : X \times Y \to R$ defined by

$$\bar{f}(x,y) = \begin{cases} 0 & \text{if } y \neq f(x), \\ 1 & \text{if } y = f(x). \end{cases}$$

Since $h' = f' \cup g'$ and f' and g' are single valued and $f'(x) = g'(x)$ for $x \in X_1 \cap X_2$, h' is single valued. Thus m-function $h : X \to Y$ defined by $h(x) = h'(x)$. Let $h(x') = y \in Y$ and $V(y)$ be an neighborhood of Y. By the definition of m-function, there is a neighborhood $\hat{V}(y)$ such that for any neighborhood $V \subset \hat{V}(y)$ there exists a nbd $U(x')$ satisfying; $\sum_{y \in V} \bar{h}(x,y) = \bar{h}(x',y) = 1$ for $x \in U(x')$. Let $V = V(y) \cap \hat{V}(y)$. Then $\bar{h}(x,y') \neq 0$ for some $y' \in V$. So $(x,y') \in h'$. But since h' is single valued, $h(x) = h'(x) = y' \in V \subset V(y)$. Thus $h(U(x')) \subset V(y)$. We conclude that h is a continuous function.

4. Proof of the main theorem

In this section, we prove the main theorem that the m-fundamental group acts on m-homotopy group as a group automorphism. In order to do that we introudce some definitons, and lemmas.

DEFINITION 4.1. A subspace A of a space X is said to have the absolute m-homotopy extension property (AMHEP) if for every m-homotopy $h : A \times I \to Y$ of an arbitrary m-function $f : X \to Y$, there exists an m-homotopy $g : X \times I \to Y$ such that $g|_{X\times 0} = f$ and $g|_{A\times I} = h$.

REMARK. A has the AMHEP in X if and only if for every defining function $\bar{h} : A \times I \times Y \to R$ and $\bar{f} : X \times Y \to R$ such that $\bar{h}(a,0,y) = \bar{f}(a,y)$ for $(a,y) \in A \times Y$, there exists a defining function $\bar{g} : A \times I \times Y \to R$ such that $\bar{g}(a,t,y) = \bar{h}(a,t,y)$ for $(a,t,y) \in A \times I \times Y$ and $\bar{g}(x,0,y) = \bar{f}(x,y)$.

LEMMA 4.2. *If (X, A) is a (finitely) triangulable pair, then A has the AMHEP in X.*

Proof. Let f be a given m-function with its defining function $\bar{f} : X \times Y \to R$ and $h : A \times I \to Y$ a given m-homotopy of f with its defining function $\bar{h} : A \times I \times Y \to R$, that is, $\bar{h}(a,0,y) = \bar{f}(a,y)$ for $a \in A, y \in Y$. Consider the product space $M = X \times I$ and its closed subspace $L = X \times 0 \cup A \times I$ Define a m-function $H : L \to Y$ by its defining function defined by

$$\bar{H}(x,0,y) = \begin{cases} \bar{f}(x,y) & \text{for } (x,0,y) \in X \times 0 \times Y \\ \bar{h}(a,t,y) & \text{for } (a,t,y) \in A \times I \times Y \end{cases}$$

We define $\bar{F} : X \times I \times Y \to R$ by $\bar{F}(x,t,y) = \bar{H}(r(x,t),y)$, where $r : X \times I \to X \times 0 \cup A \times I$ is a retraction. Then $\bar{F}(x,0,y) = \bar{H}(r(x,0),y) = \bar{f}(x,y)$ and $\bar{F}(a,t,y) = \bar{H}(r(a,t),y) = \bar{h}(a,t,y)$. Hence m-function $F : X \times I \to Y$ defined by $\bar{F}$ is an extension of h such that $F|_{X\times 0} = f$.

THEOREM 4.3. *Let $p : (I, \partial I) \to (X, x_0)$ be an m-function with multiplicity 0. Then p induces a transformation $p_n : m\pi_n(X, x_0) \longrightarrow m\pi_n(X, x_0)$ which depends only on the m-homotopy of the m-function p.*

Proof.. Let $f : (I^n, \partial I^n) \longrightarrow (X, x_0)$ be an m-function with multiplicity zero. Define $\bar{\varphi}_p : \partial I^n \times I \times X \to R$ by $\bar{\varphi}_p(u,t,x) = \bar{p}(1-t,x)$, where $\bar{p}$ is the defining function of p. Then $\bar{\varphi}_p$ defines an m-function $\varphi_p : \partial I^n \times I \to X$. Define $\bar{F}' : [(I^n \times 0) \cup (\partial I^n \times I)] \times X \to R$ by

$$\bar{F}'(u,t,x) = \begin{cases} \bar{f}(u,x) & \text{if } (u,t,x) \in I^n \times 0 \times X \\ \bar{\varphi}_p(u,t,x) & \text{if } (u,t,x) \in \partial I^n \times I \times X. \end{cases}$$

Then $\bar{F}'$ is a defining function by Theorem 3.1. So $\bar{F}'$ defines an m-function $F' : I^n \times 0 \cup \partial I^n \times I \to X$ such that $F'|_{I^n \times 0} = f$ and $F'|_{\partial I^n \times I} = \varphi_p$. By the Lemma 4.2, ∂I^n has the AMHEP in I^n. Thus there exists an m-function $F : I^n \times I \to X$ sucht that $F|_{I^n \times 0} = f$ and $F|_{\partial I^n \times I} = \varphi_p$.

From now on, we shall call such F m-funtion of f along p.

Let $F|_{I^n \times 1} = f_1$. Then $m(f_1) = m(f) = 0$ and $\bar{f}(v, x) = \bar{F}(v, 1, x) = \bar{p}(1-1, x) = 0$ for $v \in \partial I^n$, where $\bar{F}$ is a defining function of F. Define $p_n : m\pi_n(X, x_0) \to m\pi_n(X, x_0)$ by $p_n[f] = [f_1]$. Let's show that p_n is well-defined. Let $f \sim_m g(\text{rel}\partial I^n)$ and $p \sim_m q(\text{rel}\partial I)$ and H and G m-homotopies of f along p and g along q respectively. It is sufficient to show that $f_1 \sim_m g_1(\text{rel}\partial I^n)$. Since $F|_{I^n \times 0} = f$, $G|_{I^n \times 0} = g$, and $f \sim_m g(\text{rel}\partial I^n)$, there exists an m-homotopy $H_1 : I^n \times 0 \times I \to X$ such that $H_1|_{I^n \times 0 \times 0} = F|_{I^n \times 0}$, and $H_1|_{I^n \times 0 \times 1} = G|_{I^n \times 0}$, and $H_1|_{\partial I^n \times 0 \times t} = \phi$. Furthermore, since $p \sim_m q(\text{rel}\partial I)$, there is an m-function $h : I \times I \to X$ with its defining function $\bar{h} : I \times I \times X \to R$ such that $\bar{h}(t, 0, x) = \bar{p}(t, x), \bar{h}(t, 1, x) = \bar{q}(t, x)$, and $\bar{h}(\{0,1\} \times s \times x) = 0$, where $\bar{p}$ and $\bar{q}$ are the defining functions of p and q respectively. Define $\bar{H}_2 : \partial I^n \times I \times I \times X \to R$ by $\bar{H}_2(v, t, s, x) = \bar{h}(1-t, s, x)$, then

$$\bar{H}_2(v, t, 0, x) = \bar{h}(1-t, 0, x) = \bar{p}(1-t, x) = \bar{\varphi}_p(v, t, x),$$
$$\bar{H}_2(v, t, 1, x) = \bar{h}(1-t, 1, x) = \bar{q}(1-t, x) = \bar{\varphi}_q(v, t, x),$$

and

$$\bar{H}_2(v \times \{0,1\} \times s \times x) = 0.$$

Thus $\bar{H}_2$ defines an m-homotopy $H_2 : \partial I^n \times I \times I \to X$ between $F|_{\partial I^n \times I}(= \varphi_p)$ and $G|_{\partial I^n \times I}(= \varphi_q)$ relartive to $\partial I^n \times 0 \cup \partial I^n \times 1$. Let $A = I^n \times 0 \cup \partial I^n \times I$. Define $H : A \times I \to X$ by

$$H|_{I^n \times 0 \times I} = H_1 \text{ and } H|_{\partial I^n \times I \times I} = H_2.$$

Then H is well-defined m-function, because $H_1|_{\partial I^n \times 0 \times t} = \phi = H_2|_{\partial I^n \times 0 \times t}$. Since $H|_{A \times 0} = F|_A$, $H|_{A \times 1} = G|_A$, and $H|_{\partial I^n \times 0 \times t \cup \partial I^n \times 1 \times t} = \phi$, H is an m-homotopy between $F|_A$ and $G|_A$ relative to $\partial I^n \times 0 \times t \cup \partial I^n \times 1 \times t$. Thus by the AMHEP, there is an m-homotopy $H' : I^n \times I \times I \to X$ such that $H'|_{A \times I} = H$, $H'|_{I^n \times I \times 0} = F$, $H'|_{A \times 1} = G|_A$, and $H'|_{\partial I^n \times 1 \times t} = \phi$ for all $t \in I$. Let $T = H'|_{I^n \times I \times 1} : I^n \times I \to X$. Then $T|_{I^n \times 0} = g$,

$T|_{\partial I^n \times I} = \varphi_q$. So T is an m-homotopy of g along q. Let $T|_{I^n\times 1} = h_1$. Then f_1 is homotopic to h_1 relative to ∂I^n by the m-homotopy $H'|_{I^n\times 1\times I}$.

Now, Let's prove that g_1 and h_1 are m-homotopic relative to ∂I^n. Define an m-function $M : I^n \times I \to X$ by its defining function $\bar{M} : I^n \times I \times X \to R$ defined by

$$\bar{M}(u,s,x) = \begin{cases} \bar{G}(u,1-2,x) & (u \in I^n, 0 \le s \le \frac{1}{2}) \\ \bar{T}(u,2s-1,x) & (u \in I^n, \frac{1}{2} \le s \le 1) \end{cases}$$

Then for each $v \in \partial I^n$, we have $\bar{M}(v,s,x) = \bar{M}(v,1-s,x)$. Therefore we may define a m-homotopy $N : (\partial I^n \times I \cup I^n \times \partial I) \times I \to X$ by the defining function $\bar{N} : (\partial I^n \times I \cup I^n \times \partial I) \times I \times X \to R$ defined by

$$\bar{N}(u,s,t,x) = \begin{cases} \bar{M}(u,s,x) & (u \in I^n, s \in \partial I) \\ \bar{M}(u,s-ts,x) & (u \in \partial I^n, 0 \le s \le \frac{1}{2}) \\ \bar{M}(u,(1-s)(1-t),x) & (u \in \partial I^n, \frac{1}{2} \le s \le 1) \end{cases}$$

Since $\partial I^n \times I \cup I^n \times \partial I$ has the AMHEP in $I^n \times I$, the m-homotopy N has an extension m-homotopy $L : I^n \times I \times I \to X$ such that $L|_{I^n\times I\times 0} = M$. Let $O = L|_{I^n\times I\times 1}$. Then $O|_{I^n\times 0} = L|_{I^n\times 0\times 1} = N|_{I^n\times 0\times 1} = M|_{I^n\times 0} = G|_{I^n\times 1} = g_1, O|_{I^n\times 1} = T|_{I^n\times 1} = h_1$, and $O|_{\partial I^n\times s} = \phi$ for every $s \in I$. This implies that g_1 and h_1 are m-homotopic relative to ∂I^n.

So we have constructed a transformation $p_n : m\pi_n(X) \to m\pi_n(X)$ which depends only on the m-homotopy of the m-function $p : (I, \partial I) \to (X, x_0)$.

THEOREM 4.4. *$m\pi_1(X)$ acts on $m\pi_n(X)$ as a group automorphism, $n \ge 1$.*

Proof. It is sufficient to show that p_n constructed at the Theorem 4.3 is isomorphism.

Let α, β be arbitrary elememts of $m\pi_n(X)$ represented by the m-functions with multiplicity zero $f, g : (I^n, \partial I^n) \to (X, x_0)$. Let $F, G : I^n \times I \to X$ be m-homotopies along p of f, g respectively. Then $f_1 = F|_{I^n\times 1}$ represents $p_n(\alpha)$ and $g_1 = G|_{I^n\times 1}$ represents $p_n(\beta)$. Let $\overline{F\cdot G} : I^{n-1} \times I \times I \times X \to R$ be the defining function defined by

$$\overline{F\cdot G}(u,s,t,x) = \begin{cases} \bar{F}(u,2s,t,x) & 0 \le s \le \frac{1}{2} \\ \bar{G}(u,2s-1,t,x) & \frac{1}{2} \le s \le 1. \end{cases}$$

Then $\overline{F \cdot G}$ defines an m-function $F \cdot G : I^n \times I \to X$. But

$$\overline{F \cdot G}(u, s, 0, x) = \begin{cases} \bar{F}(u, 2s, 0, x) & 0 \le s \le \frac{1}{2} \\ \bar{G}(u, 2s-1, 0, x) & \frac{1}{2} \le s \le 1 \end{cases}$$
$$= \begin{cases} \bar{f}(u, 2s, x) & 0 \le s \le \frac{1}{2} \\ \bar{g}(u, 2s-1, x) & \frac{1}{2} \le s \le 1, \end{cases}$$

where $\bar{f}, \bar{g}$ are defining functions of f, g respectively. So $F \cdot G|_{I^n \times 0} = f \cdot g$. On the other hand, for $(u, s) \in \partial(I^{n-1} \times I) = \partial I^{n-1} \times I \cup I^{n-1} \times \{0, 1\}$, $\overline{F \cdot G}(u, s, t, x) = \bar{p}(1-t, x)$, where $\bar{p}$ is a defining function of p. So $p_n[f \cdot g] = [F \cdot G|_{I^n \times 1}] = [f_1 \cdot g_1]$. Since $fg \sim_m f + g(\text{rel}\partial I^n)$ by Theorem 2.2

$$p_n[f+g] = p_n[f \cdot g] = [f_1 \cdot g_1] = [f_1 + g_1] = [f_1] + [g_1] = p_n[f] + p_n[g].$$

Consequently, p_n is a homomorphism.

Finally, let us prove that the homomorphism p_n is an isomorphism. First we show that the composition of $p_n \circ q_n = (p+q)_n$, where $p, q : (I, \partial I) \to (X, x_0)$ are m-functions with multiplicity zero. Let $q_n[f] = [f_1]$ and $p_n[f_1] = [f_2]$. Then there are m-homotopies F_1, F_2 of f, f_1 along p, q respectively. This means that there are defining functions $\bar{F}_1, \bar{F}_2 : I^n \times I \times X \to R$ such that $\bar{F}_1(u, 0, x) = \bar{f}(u, x), \bar{F}_1(v, t, x) = \bar{q}(1-t, x)$ for $v \in \partial I^n$, and $\bar{F}_1(u, 1, x) = \bar{f}_1(u, x)$, and $\bar{F}_2(v, t, x) = \bar{p}(1-t, x)$ for $v \in \partial I^n$, $\bar{F}_2(u, 0, x) = \bar{f}_1(u, x)$, $\bar{F}_2(u, 1, x) = \bar{f}_2(u, x)$. Define the defining function $\bar{F} : I^n \times I \times X \to R$ by taking

$$\bar{F}(u, t, x) = \begin{cases} \bar{F}_1(u, 2t, x) & 0 \le t \le \frac{1}{2} \\ \bar{F}_2(u, 2t-1, x) & \frac{1}{2} \le t \le 1. \end{cases}$$

Then $\bar{F}(u, 0, x) = \bar{f}(u, x)$, $\bar{F}(v, t, x) = \begin{cases} \bar{q}(1-2t, x) & 0 \le t \le \frac{1}{2} \\ \bar{p}(2-2t, x) & \frac{1}{2} \le t \le 1. \end{cases}$

Furthermore, $F(u, 1, x) = \bar{f}_2(u, x)$. So $\bar{F}$ defines a m-homotopy $F : I^n \times I \to X$ such that $F_{I^n \times 0} = f, F|_{I^n \times 1} = f_2$, and $F|_{\partial I^n \times I} = \varphi_{p \cdot q}$. This means $(p \cdot q)_n[f] = [f_2]$. Since $p \cdot q \sim_m p + q(\text{rel}\partial I)$ by Theorem 2.2, $(p \cdot q)_n = (p+q)_n$ by Theorem 4.3. Thus

$$p_n(q_n[f]) = p_n[f_1] = [f_2] = (p \cdot q)_n[f] = (p+q)_n[f].$$

Let $\mathcal{O} \in m\pi_1(X)$ be the 0-element. Then $\mathcal{O}n[f] = [f]$ for every $[f] \in m\pi_n(X)$. In fact, if $F : I^n \times I \to X$ is a m-homotopy of f along $\mathcal{O}$, then $F|_{I^n\times 0} = f$, $F|_{I^n\times 1} = f_1$, and $F|_{\partial I^n \times I} = \varphi_{\mathcal{O}} = \phi$. So $f \sim_m f_1(\mathrm{rel}\partial I^n)$. Since

$$p_n(-p_n)[f] = (p-p)_n[f] = \mathcal{O}_n[f] = [f],$$

p_n is an epimophism. Moreover, since $(-p_n)(p_n[f]) = (-p+p)_n = \mathcal{O}_n[f] = [f], p_n$ is a monomorphism. We conclude p_n is an isomorphism.

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