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NONLINEAR TANGENTIAL OBLIQUE BOUNDARY VALUE PROBLEM

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1. Introduction

Let Ω be a bounded and smooth domain in $R^n, n \geq 2$. It is well known that the problem of finding a function $u \in C^\infty(\bar{\Omega})$ satisfying the relations

$$Lu = 0 \text{ in } \Omega$$
$$\frac{\partial u}{\partial l} = f \text{ on } \partial\Omega$$

is ill-posed if l is tangential to $\partial\Omega$. In [5], B.Winzell studied this problem for the uniformly elliptic linear second order operators of the following form;

$$Lu = a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u.$$

In [4], P.R.Popivanov and N.D.Kutev studied this problem for the operator L of the following form;

$$Lu = a^{ij}(x)D_{ij}u + b^i(x,u)D_iu + c(x,u).$$

However the linear principal part of the above equation results from the key idea of Egorov and Knodratiev to differentiate the equation in the tangential direction l and is not required essentially by the problem.

In this paper we study the same problem for the quasilinear uniformly elliptic operator

$$Qu = a^{ij}(x,u,\nabla u)D_{ij}u + a(x,u,\nabla u).$$

Here a^{ij} and a are given functions of their arguments. The most widely studied such problem is the Dirichlet problem;

$$Qu = 0 \text{ in } \Omega$$
$$u = \varphi \text{ on } \partial\Omega.$$

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THEOREM A. *There are various conditions under which the Dirichlet problem*

$$(1.1)\qquad \begin{cases} Qu = a^{ij}(x, u\nabla u)D_{ij}u + a(x, u, \nabla u) = 0 \text{ in } \Omega \\ u = \varphi \text{ on } \partial\Omega \end{cases}$$

is solvable for any $\varphi \in C^\infty(\partial\Omega)$.

For the various conditions that guarantee the solvability of the Dirichlet problem (1.1), we refer to the extensive literature (in particular [1]). A more general boundary value problem is that in which the gradient of the solution plays a role in the boundary condition

$$Nu = 0 \text{ on } \partial\Omega,$$

where N is given on $C^1(\bar{\Omega})$ by

$$Nu(x) = b(x, u(x), \nabla u(x)).$$

If N is assumed to satisfy a condition analogous to the ellipticity of Q, there are many existence theorems [2]. However if the boundary operator N is everywhere tangential, then the study of boundary behaviour of the solution is essentially separate from the existence problem. From now on we consider only the operators Q which satisfy one of the solvability conditions of Theorem A.

2. Main Result

In this section, we consider the following oblique derivative problem

$$(2.1)\qquad \begin{cases} Qu = a^{ij}(x, u\nabla u)D_{ij}u + a(x, u, \nabla u) = 0 \text{ in } \Omega \\ \frac{\partial u}{\partial l} = f \text{ on } \partial\Omega \end{cases}$$

Here l is a C^∞ tangent vector field on $\partial\Omega$ and f is a given C^∞ function on $\partial\Omega$. In order to solve the problem (2.1), by Theorem A, it suffices to find a function $\phi \in C^\infty(\partial\Omega)$ such that

$$(2.2)\qquad \frac{\partial u}{\partial l} = f.$$

A necessarily condition for the solvability of the problem (2.2) is that

$$(2.3)\qquad \int_c f = 0$$

for any closed integral curve C of l.

LEMMA 2.1. *Ω is a bounded smooth domain in R^2 and $\partial\Omega$ is a connected C^∞ manifold. Let l be a C^∞ tangent vector field to $\partial\Omega$ which never vanishes. Then (2.1) is solvable for any f satisfying the condition (2.3).*

Proof. Observe that any point $p \in \partial\Omega$ has a neghborhood U_p and a function $g_p \in C^\infty(U_p)$ such that

$$\frac{\partial g_p}{\partial l} = f \text{ on } U_p.$$

We can cover $\partial\Omega$ by a finite number of open set $U_1, U_2, \cdots, U_k$ with corresponding functions $g_1, g_2, \cdots, g_k$ such that $U_i \cap U_j \neq 0$ if and only if $j = i+1 \pmod k$. Choose constants c_i's so that

$$\begin{aligned}
&\varphi_i = \varphi_{i+1} \text{ on } U_i \cap U_{i+1}, i = 1, 2, \cdots, k-1,\\
&\varphi_1 = f_1,\\
&\varphi_i = f_i + c_{i-1}, i = 2, 3, \cdots, k-1.
\end{aligned}$$

Then condition (2.3) implies that

$$\varphi_k = \varphi_1 \text{ on } U_k \cap U_1.$$

Therefore we can glue together $\varphi_1, \varphi_2, \cdots, \varphi_k$ to a single function $\varphi \in C^\infty(\partial\Omega)$ with $\frac{\partial\varphi}{\partial l}$.

Now by theorem A we can solve the Dirichlet problm $Qu = 0$ in Ω and $u = \varphi$ on $\partial\Omega$. This completes the proof.

Since $\partial\Omega$ is compact, l is complete. Let $\Psi : R \times \partial\Omega \to \partial\Omega$ be a 1-parameter group of transformations of $\partial\Omega$ for l.

COROLLARY 2.2. *Let Ω be a bounded smooth domain in $R^n, n \geq 3$. l is a tangent vector field to $\partial\Omega$ which never vanishes. Suppose there exists a $(n-2)$-dimensional connected submanifold D of $\partial\Omega$ such that each integral curves of l starting from $p \in D$ retruns to p in a finite time. Suppose l is transversal to D and*

$$\bigcup_{(t,p)\in R\times D} \psi_p(t) = \partial\Omega.$$

Then (2.1) is solvable for any f satisfying the condition (2.3).

Proof. Choose any function $h \in C^\infty(\partial\Omega)$. Define a function $\phi : \partial\Omega \longrightarrow R$ by the following procedure; Given $x \in \partial\Omega$, choose $p \in \partial\Omega$ such that $x = \psi_p(t)$ for some $t \in R$. By applying the method of proof of the above Lemma to the orbit of l through p, we get a funcion g_p defined on the orbit with $g_p(p) = h(p)$. Take $\psi(x) = g_p(x)$. Then $\frac{\partial\varphi}{\partial l} = f$ by our construction. Now theorem A completes the proof.

Again we denote by $\Psi : R \times \partial\Omega \longrightarrow \partial\Omega$ the 1-parameter group of transformations of $\partial\Omega$ for l.

THEOREM 2.3. *Let Ω be a bounded and smooth domain in R^3. Suppose $\partial\Omega$ is connected and every simple closed curve in $\partial\Omega$ separates $\partial\Omega$ into two components. Suppose that the vector field l satisfies the following conditions;*

(i) there exist a 1-dimensional path connected submanifold with boundary D such that

$$\bigcup_{(t,p)\in R\times D} \{\Psi(t,p)\} = \partial\Omega,$$

(ii) l vanishes only on ∂D with indices 1,
(iii) every integral curve of l is of finite length.

Then (2.1) has a solution for every f which vanishes on ∂D and which satisfies the condition (2.3).

Proof. Let C_p denote the orbit of l through $p \in D$. Conditions (ii) and (iii) implies that C_p is a simple closed curve in $\partial\Omega$. Hence $C_p \cap D$ is a finite set. If $C_p \cap D$ contains more then one point, there exists t_1 such that $\Psi(t_1, p) \in D$ but $\Psi(t_1, p) \notin D$ for $0 < t < t_1$. Let $q = \Psi(t_1, p)$. Choose a path $\alpha : [0,1] \longrightarrow D$ from q to p. Define $\beta : [0, t_1 + 1] \longrightarrow \partial\Omega$ by

$$\beta(s) = \begin{cases} \Psi(s,p), & 0 \le s \le t_1, \\ \alpha(s - t_1), & t_1 \le s \le t_1 + 1. \end{cases}$$

Then β is a simple closed curve separating $\partial\Omega$ into two components. But it contradicts the fact that C_p is a simple closed curve. Hence

$C_p \cap D = \{p\}$. Hence for each $x \in \partial\Omega$ there exist a unique point $p \in D$ such that $x \in C_p$. Now Corollary 2.2 completes the proof.

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