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CONTINUOUS RINGS WHOSE CERTAIN HOMOMORPHIC IMAGES ARE GOLDIE

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It is well known by Faith [6] that a right self-injective ring with ACC (ascending chain condition) on essential right annihilators is a quasi-Frobenius (simply QF) ring. Thus for self-injective rings, a restricted form of chain conditions may imply the maximum (or minimum) condition. In fact, in [3] Armendariz proved that a right self-injective ring is QF if it satsfies DCC (descending chain condition) on essential right ideals. This result has been extended to the case when such a ring has ACC on essential right ideals by [5]. Furthermore all these results have been generalized by Armendariz and Park [4] to the case of a right self-injective ring whose factor ring modulo the right socle is right Goldie.

By hiring some useful techniques developed in [2 and 4], we study in this paper continuous rings with some types of restricted chain conditions related to results in [1,2, 3 and 4]. Actually we prove that a right continuous ring whose factor ring modulo the right socle is right Goldie is a semiprimary ring with finitely generated right socle. Thereby we can get the result of Jain, Lopez-Permouth and Rizvi [7] as an immediate consequence; that is, a right continuous ring with ACC on essential right ideals is a right Artinian ring.

Recall that a ring R is called *right* (resp.*left*) *continuous* provided every right (resp.left) ideal is essential in a direct summand of R and a right (resp.left) ideal isomorphic to a direct summand is generated by an idempotent.

Obviously, every right (resp.left) self-injective ring is right (resp.left) continuous. But there is a continuous ring which is not self-injective. In fact, let $F[x_1, x_2, ...]$ be the polynomial ring over a field F with countably infinite undeterminates $x_1, x_2, ...$ Now in the ring $F[x_1, x_2, ...]$, let I be the ideal generated by the elements of the following types;

$$x_i x_j x_k, x_n x_{n+1} - x_1 x_2, x_p x_q,$$

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where i, j, k, n, p, and q are positive integers with $| p - q | \neq 1$. Then the ring

$$R = F[x_1, x_2, ...]/I$$

is a continuous ring but not self-injective.

For a ring R, the *right singular ideal* $Z_r(R)$ is the set of all elements in R whose right annihilators are essential in R as right R-modules.

We start with following

LEMMA 1. *[9, Utumi]. Assume that a ring R is right continuous. Then we have following:*

(1) The ring R/J is a von Neumann regular right continuous ring, where J is the Jacobson radical of R.

(2) $J = Z_r(R)$.

(3) Every set of orthogonal idempotents of the ring R/J can be lifted to a set of orthogonal idempotents of R.

A ring R is said to be *right Goldie* if (1) R satisfies ACC on right annihilators, and (2) R contains no infinite direct sums of non-zero right ideals. Every right Noetherian ring is of course a right Goldie ring. But the polynomial ring over a field with infinitely many commuting undeterminates is right Goldie but not right Noetherian.

It is a standars fact that every nil one-sided ideal of a right Noetherian ring is nilpotent. In the following, this fact is generalized to the case of Goldie rings.

LEMMA 2. *[8, Lanski]. Every nil one-sided ideal of a ring Goldie ring is nilpotent.*

Finally, for our preparations, recall that a ring R is called *semiprimary* if R/J is Artinian and J is nilpotent, where J is the Jacobson radical of R.

THEOREM 3. *Assume that a ring R is right continuous and the ring $R/Soc_r(R)$ is right Goldie. Then we have following:*

(1) $Soc_r(R)$ is right Artinian as a right R-module.

(2) R is semiprimary.

Proof. (1) Since the ring $R/Soc_r(R)$ is right Goldie, it is finite Goldie dimensional. Now say k is the right Goldie diemnsion of $R/Soc_r(R)$. To prove that $Soc_r(R)$ is right Artinian, actually we need to show that it is finitely generated as a right R-module.

Assume to the contrary that $Soc_r(R)$ is not finitely generated as a right R-module. Then of course $Soc_r(R)$ is an infinite direct sum of minimal right ideals of R. Thus we can write $Soc_r(R) = \bigoplus_{i \in I} A_i$ with I an infinite set and each A_i a minimal right ideal of R. So we can take a partition of I as $I = I_1 \cup I_2 \cup ... \cup I_k \cup I_{k+1}$(disjoint union) with $|I_1| = |I_2| = ... = |I_{k+1}|$. Note that in this case for each $i = 1, 2, ..., k+1$, the set I_i is an infinite set. Let $S_i = \bigoplus_{j \in I_i} A_j$ for each $i = 1, 2, ..., k+1$. Then $Soc_r(R) = S_1 \oplus S_2 \oplus ... \oplus S_{k+1}$ and each S_i is not finitely generated. By assumption, since R is right continuous, there is an idempotent e_i of R such that S_i is essential in e_iR as a right R-module for $i = 1, 2, ..., k+1$. Thus we have $S_1 \oplus S_2 \oplus ... \oplus S_{k+1}$ is essential in $e_1R \oplus e_2R \oplus ... \oplus e_{k+1}R$. But since S_i is not finitely generated, we have $S_i \neq e_iR$ and thus $e_iR/S_i \neq 0$ for $i = 1, 2, ..., k+1$. So we have

$$\frac{e_1R \oplus e_2R \oplus ... \oplus e_{k+1}R}{S_1 \oplus S_2 \oplus ... \oplus S_{k+1}} \subseteq \frac{R}{Soc_r(R)}.$$

That is, $e_R/S_1 \oplus e_2R/S_2 \oplus ... \oplus e_{k+1}R/S_{k+1} \subseteq R/Soc_r(R)$ with $e_iR/S_i \neq 0$ for $i = 1, 2, ..., k+1$. Hence the right Goldie dimension of $R/Soc_r(R)$ is greater than or equal to $k+1$. This is a contradiction. Thus $Soc_r(R)$ is finitely generated.

(2) We show at first that the Jacobson radical of R is nilpotent. Let a be an element of J. Then since $J = Z_r(R)$, the right singular ideal of R by Lemma 1, we have that $r(a)$, the right annihilator of a in R, is an essential right ideal of R. Therefore $Soc_r(R) \subseteq r(a)$ because the right socle $Soc_r(R)$ is the intersection of all essential right ideals of R. So we have following series of right ideals:

$$Soc_r(R) \subseteq r(a) \subseteq r(a^2) \subseteq r(a^3) \subseteq \cdots .$$

Now for any positive integer k, we have

$$r(a^k)/Soc_r(R) \subseteq r(a^k + Soc_r(R)) \subseteq r(a^{k+1})/Soc_r(R),$$

where $r(a^k + Soc_r(R))$ denotes the right annihilator of $a^k + Soc_r(R)$ in the ring $R/Soc_r(R)$. In fact the inclusion $r(a^k)/Soc_r(R) \subseteq r(a^k + Soc_r(R))$

is obvious. For the other inclusion, let $b+Soc_r(R)$ be in $r(a^k+Soc_r(R))$. Then we have $a^kb \in Soc_r(R)$. Therefore $a^kb \in r(a)$ and hence $a^{k+1}b=0$, that is, $b \in r(a^{k+1})$. So $b+Soc_r(R)$ is in $r(a^{k+1})/Soc_r(R)$. By the given assumption, $R/Soc_r(R)$ is right Goldie and hence there is a positive integer t such that $r(a^t+Soc_r(R)) = r(a^{t+1}+Soc_r(R)) = \cdots$. But note that $r(a^{t+1})/Soc_r(R)$ is inserted between $r(a^t+Soc_r(R))$ and $r(a^{t+1}+Soc_r(R))$. Thus we have that $r(a^{t+1})/Soc_r(R) = r(a^{t+2})/Soc_r(R) = \cdots$. Let $N=t+1$. Then of course we have $r(a^N)=r(a^{N+1})=\cdots$.

For a while, our claim is that $r(a) \cap a^NR = 0$. Indeed, if y is in $r(a)\cap a^NR$, then $ay=0$ and $y=a^Nx$ for somem x in R. Thus $0=ay=a^{N+1}x$ and so x is in $r(a^{N+1})=r(a^N)$. Therefore $y=a^Nx=0$.

Since $r(a)$ is an essential right ideal of R, we jhave $a^NR=0$ from our claim $r(a)\cap a^NR=0$. Thus $a^N=0$. By this fact the ideal $(J+Soc_r(R))/Soc_r(R)$ is nil. By Lemma 2, every nil one-sided ideal of a right Goldie ring is nilpotent. Therefore the ideal $(J+Soc_r(R))/Soc_r(R)$ is nilpotent. Thus there is a positive integer m such that $J^m \subseteq Soc_r(R)$. So we have $J^{m+1} \subseteq Soc_r(R)J=0$ and hence J is nilpotent.

Now to prove that R is semiprimary, note that by Theorem 1 R/J is von Neumann regular and every set of orthogonal idempotents of R/J can be lifted to a set of orthogonal idempotents of R. Now we claim that R/J is Artinian. For, if it is not Artinian, then there is a set of infinitely many orthogonal idempotents $\{f_1, f_2, ...\}$ of R/J. So this set of orthogonal idempotents can be lifted to a set of orthogonal idempotents $\{e_1, e_2, ...\}$ of R. In this situation, since $Soc_r(R)$ can contain only finitely many orthogonal idempotents, there is a positive integer N such that $e_1, e_2, ..., e_{N-1}$ can be only contained in $Soc_r(R)$. Thus $\{e_N+Soc_r(R), e_{N+1}+Soc_r(R), ...\}$ forms a set of infinitely many nonzero orthogonal idempotents in the right Goldie ring $R/Soc_r(R)$. But this is impossible. Hence R/J is Artinian and so R is semiprimary.

By Theorem 3, we can get the following result immediately.

COROLLARY 4. *[7, Jain, Lopez-Permouth and Rizvi]. If a right continuous ring R satisfies ACC on essential right ideals, then R is right Artinian.*

Proof. By [5] $R/Soc_r(R)$ is right Noetherian. So R is right continuous with $R/Soc_r(R)$ right Goldie. Thus by Theorem 3, R is semiprimary and $Soc_r(R)$ is finitely generated. Therefore R is right Noetherian and semiprimary. So R is right Artinian.

Also as a byproduct of Corollary 4, we have following

COROLLARY 5. *If a right self-injective ring R satisfies ACC on essential right ideals, then R is QF.*

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