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ESSENTIAL SUBALGEBRAS OF LIE ALGEBRA

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1. Introduction

In this note we discuss an essential subalgebra of a finite dimensional filtered Lie algebra of characteristic $p > 2$. By a filtration of length r of a simple Lie algebra L is meant a sequence of subalgebras

$$L = L_{(-1)} \supset L_{(0)} \supset \cdots \supset L_{(r)} \supset L_{(r+1)} = (0),\ L_{(r)} \neq (0),$$

where $L_{(0)}$ is a given non-zero proper subalgebra and the remaining algeras are inductively defined by

$$L_{(i+1)} = \{x \in L_{(i)} | xL \subset L_{(i)}\} \text{ for } 1 \leq i \leq s.$$

If r is a maximal length of all filtrations of L, then r is called the length of filtration of L. And if a Lie algebra has a finite length of filtration, then L is called a filtered Lie algebra. The generalized Jacobson-Witt algebra $W(n;\underline{k})$ is a filtered Lie algebra, where $n \geq 1$ and $\underline{k} = (k_1, k_2, \cdots, k_n)$ with each $k_i \geq 1$.

2. Essential Subalgebras

The purpose of this section are to show that (1) the essential subalgebra of a given restricted Lie algebra L has the filtration which is induced from the one of L and (2) if S is an essential subalgebra of $W(1;\underline{1})$, then either S coincides with $W(1;\underline{1})$ or there exists an embedding of S into $\mathrm{sl}(2,F)$. Also we develop (2) into the case of Zassenhaus algebra $W(1;\underline{k})$

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DEFINITION 2.1. Let L be a finite dimensional Lie algebra over F of characteristic p. And $L_{(0)}$ be a subalgebra of L. Then S is called an essential subalgebra of L with respect to $L_{(0)}$ if

(i) $S + L_{(0)} = L$ and

(ii) $S \cap L_{(0)}$ contains no proper nonzero ideals of S.

We proved in [2], for any restricted Lie algebra L, there is a homomorphism φ from L into the Jacobson-Witt algebra $W(n;\underline{1})$. If L is simple, then φ is a monomorphism and $\varphi(L)$ is an essential subalgebra of $W(n;\underline{1})$ as following:

THEOREM 2.2. *[4] Let $(L,[p])$ be a restricted Lie algebra and suppose that $L_{(0)}$ is a p-subalgebra of codimension n. The mapping $\varphi : L \to W(n;\underline{1})$ enjoys the following properties:*

(i) $\ker\varphi \subset L_{(0)}$ is an unique maximal ideal of L which is contained in $L_{(0)}$.

(ii) $\varphi(L_{(0)}) = W(n;\underline{1})_{(0)} \cap \varphi(L)$ and $\varphi^{-1}(W(n;\underline{1})_{(0)}) = L_{(0)}$.

(iii) $\dim_F(\varphi(L)/W(n;\underline{1})_{(0)} \cap \varphi(L)) = n$.

(iv) $\varphi(L)$ is an essential subalgebra of $W(n;\underline{1})$.

COROLLARY 2.3. *Let $(L,[p])$ be a restricted simple Lie algebra of dimension np^n and $L_{(0)}$ a p-subalgebra of codimension n. Then $\varphi : L \to W(n;\underline{1})$ is an isomorphism which satisfies $\varphi(L_{(0)}) = W(n;\underline{1})_{(0)}$.*

Proof. By the previous theorem, φ is a homomorphism with the properties (i) $\sim$ (iv). By the simplicity of L and (i) in the previous theorem, we have that $\ker\varphi = 0$ and $\dim\varphi(L) = \dim L = np^n \le \dim W(n;\underline{1}) = np^n$. Thus $\mathrm{Im}\varphi = W(n;\underline{1})$.

Let $L_{(0)}$ be a proper subalgebra of an arbitrary filtered Lie algebra L and $\{L_{(i)} | i \in Z\}$ a filtration of L defined by $L_{(0)}$, where $L_{(-1)} = L$ and $L_{(i+1)} = \{x \in L_{(i)} | [x, L] \subset L_{(i)}\}$ for $1 \ge 0$. The following lemma shows that in the case of an essential subalgebra, its filtration defined by an essential subalgebra $L_{(0)}$ coincides with the one inherited from $W(n;1)$. First, we state Lemma 2.2 in [4] and we will prove a similar result in the more general case.

LEMMA 2.4. *[4] Suppose that $S \subset W(n;\underline{1})$ is an essential with respect to $S \cap W(n;\underline{1})_{(0)} \ne 0$ and let $S_{(0)} = S \cap W(n;\underline{1})_{(0)}$. And $\{S_{(i)}\}_{i\in H}$ denotes a filtration of S defined by $S_{(0)}$. Then $S_{(i)} = S \cap W(n;\underline{1})_{(i)} \forall_i \ge 0$.*

Proof. Use the induction of i. By hypothesis, it is true when $i = 0$. Since $[S \cap W(n;\underline{1})_{(i+1)}, S] \subset S \cap W(n;\underline{1})_{(i)}$, $S \cap W(n;\underline{1}) \subset S_{(i+1)}$. On the other hand, $S_{(i+1)} \subset S_{(i)} \subset S \cap W(n;\underline{1})_{(i)}$. Let $x \in S_{(i+1)}$. By the definition of essentiality,

$$[x, W(n;\underline{1})] \subset [x, W(n;\underline{1})_{(0)}] + [x, S] \subset W(n;\underline{1})_{(i)} + S_{(i)} \subset W(n;\underline{1})_{(i)}$$

Therefore, x is an element in $W(n;\underline{1})_{(i+1)}$.

THEOREM 2.5. *Let $L_{(0)}$ be a proper subalgebra of a restricted, finite dimensional Lie algebra L and $\{L_{(i)}|i \in H\}$ be the filtration of L defined by $L_{(0)}$. Let S be an essential subalgebra with respect to $L_{(0)}$. Then S has a filtration $\{S_{(i)}|i \in H\}$ defined by $S_{(0)} = L_{(0)} \cap S$. We also have $S_{(i)} = L_{(i)} \cap S$ for $i > 0$.*

Proof. For the general case, we also use an induction on i. If $i = 1$, let $x \in S \cap L_{(1)}$. Then $x \in S$ implies $[x, S] \subset S$ and $x \in L_{(1)}$ implies $[x, L] \subset L_{(0)}$. So from $[x, S] \subset S$ and $[x, S] \subset [x, L] \subset L_{(0)}$, we have $[x, S] \subset S \cap L_{(0)} = S_{(o)}$. By definition, $S_{(1)} = \{x \in S_{(0)}|[x, S] \subseteq S_{(0)}\}, x \in S_{(1)}$ and $L_{(1)} \cap S \subset S_{(1)}$. Conversely, if $x \in S_{(1)}$, then $[x, S] \subset S_{(0)}$. Since $x \in S_{(1)} \subset S_{(0)} = S \cap L_{(0)} \subset L_{(0)}, [x, L_{(0)}] \subset L_{(0)}$,

$$[x, L] = [x, (L_{(0)} + S)] = [x, L_{(0)}] + [x, S] \subset L_{(0)} \cap S_{(0)} \subset L_{(0)}.$$

So $x \in L_{(1)}$, and $S_{(1)} \subset L_{(1)}$. Therefore, $S_{(1)} = S \cap L_{(1)}$. Assume that $S_{(i)} = S \cap L_{(i)}$ is true for i and $S_{(i+1)} \neq 0$. Let $x \in L_{(i+1)} \cap S$. Then $x \in L_{(i+1)}$ and $x \in S$ imply that $[x, L] \subset L_{(i)}$. Also $[x, S] \subset L_{(i)}$ and $[x, S] \subset S$ are hold. By the induction hypothesis, for some i,

$$[x, S] \subset S \cap L_{(i)} = S_{(i)}.$$

So $x \in S_{(i+1)}$, and $L_{(i+1)} \cap S \subset S_{(i+1)}$. Conversely, let $x \in S_{(i+1)}$. Then $[x, S] \subset S_{(i)}$. Since $x \in S_{(i)}$ implies $x \in L_{(i)}$ and it implies $[x, L_{(0)}] \subset [L_{(i)}, L_{(0)}] \subset L_{(i)}$,

$$[x, L] = [x, (L_{(0)} + S)] = [x, L_{(0)}] + [x, S] \subset L_{(i)} + S_{(i)} \subset L_{(i)}.$$

Thus $x \in L_{(i+1)}$ and $x \in S$ imply that $x \in L_{(i+1)} \cap S$. Therefore $S_{(i+1)} \subset L_{(i+1)} \cap S$, and $S_{(i+1)} = L_{(i+1)} \cap S$.

THEOREM 2.6. *Suppose that S is an essential subalgebra of $W(1;\underline{1})$. Then either S coincides with $W(1;\underline{1})$ or there exists an embedding of S into $sl(2,F)$.*

Proof. Consider the filtration of S defined by $S_0 = S \cap W(1;\underline{1})_{(0)}$. Since $W(1;\underline{1})_{(0)} = \langle \{e_i | i \geq 0\} \rangle$ and S is essential, e_{-1} is contained in S. If S contains e_i for some $i \geq 2$, then $[e_{-1}, e_i] = (i+1)e_{i-1}$ is in S, and $\{e_{-1}, e_i\}$ in S. By repeated multiplication by e_{-1} and e_i, we get a set $\{e_{-1}, e_0, e_1, \cdots, e_{i-1}, e_i, e_{i+1}, \cdots, e_{p-2}\}$ in S. Therefore, $S = W(1;\underline{1})$. Otherwise, if 1 is the largest integer i such that $e_i \in S$, then S must be a subalgebra generated by $\{e_{-1}, e_0, e_1\}$. So S has dimension 3, and $S \simeq \mathrm{sl}(2,F)$. If 0 is the largest integer such that $e_0 \in S$, then $S = \langle e_{-1}, e_0 \rangle$ and it is isomorphic into $\mathrm{sl}(2,F)$. Also if $S \cap W(1;\underline{1})_{(0)} = \emptyset$, then $S = \langle e_{-1} \rangle$, and it is embedded into $\mathrm{sl}(2,F)$.

COROLLARY 2.7. *If S is a simple subalgebra of $W(1;\underline{k})$ containing e_{-1}, then S is essential in $W(1;\underline{k})$.*

Proof. The result is induced from the definition of essentiality, because $S + W(1;\underline{k})_{(o)} \supset \langle e_{-1} \rangle + W(1;\underline{k})_{(0)} = W(1;\underline{k})$.

COROLLARY 2.8. *In case of $W(1;\underline{k})$, let S be an essential subalgebra with respect to $W(1;\underline{k})_{(0)}$. Then either S coincides with $W(1;\underline{k})$ or there exists an embedding of S into $sl(2,F)$.*

Proof. The proof is similar to the proof of Theorem 2.6.

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