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AN INVARIANCE PRINCIPLE FOR WEAKLY ASSOCIATED RANDOM VECTORS

TAE SUNG KIM AND DAE HEE RYU

1. Introduction.

A finite collection $\{X_1, \cdots, X_n\}$ of random variables is said to be associated if for any two coordinatewise nondecreasing functions f_1, f_2, on R^n such that $\tilde{f}_i = f_i(X_1, \cdots, X_n)$ has finite variance for $i = 1, 2$ there holds $\mathrm{Cov}(\tilde{f}_1, \tilde{f}_2) \geq 0$. An infinite collection is associated if every finite subcollection is associated (cf. Esary, Proschan and Walkup(1967)).

Under some covariance restrictions a wide number of limit theorems for associated sequence have been investigated. In stationary case, Newman(1980) proved the central limit theorem, Newman and Wright(1981) extended this to an invariance principle, Wood(1983) gave an extimate for the rate of uniform convergence and finally Dabrowski(1985) proved a functional law of the iterated logarithm. Cox and Grimmett(1984) extended Newman's result [8] to nonstationary case and Birkel(1988) proved an invariance principle for nonstationary associated processes. In addition, Burton, Dabrowski, and Dehling(1986) defined weakly associated random vectors and proved an invariance principle for strictly stationary sequence of weakly associated random vectors by extension of the Cramer-Wold device to suit the special needs of weakly associated random vectors, and Dabrowski and Dehling(1986) proved a Berry Esseen Theorem and a functional law of the iterated logarithm for a strictly stationary weakly associated random vectors.

The purpose of this paper is to make up a multivariate version of central limit theorem of Cox and Grimmett(1984) and to extend an invariance principle of Burton et al.(1986) to nonstationary weakly associated random vectors.

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In Section 2 by an application of Cramer-Wold technique to the central limit theorem of Cox and Grimmett [4] we make up a central limit theorem for nonstationary weakly associated reandom vectors. In Section 3 we obtain an extended Newman's inequality and extend Theorems 2.2 and 2.3 to an invariance principle for weakly associated random vectors(Theorem 3.1).

2. A Central Limit Theorem

Weak association is analogous to a positive version of negative association as well as a weakened version of association(see Newman[9]) and it defines a strictly larger class than does association(e.g. A-compound process of Burton et al.(1986)).

DEFINITION 2.1. (Burton et al.,1986) A finite collectiion $\{X_1, \cdots, X_m\}$ of R^d-valued random vectors is said to be weakly associated if whenever π is a permutation of $\{1, 2, \cdots, m\}, 1 \leq k < m$ and $f : R^{kd} \to R$, $g : R^{(m-k)d} \to R$ are coordintewise nondecreasing functions

$$\mathrm{Cov}(f(X\pi_{(1)}, \cdots, X\pi_{(k)}), g(X\pi_{(k+1)}, \cdots, X\pi_{(m)})) \geq 0.$$

An infinite collection of R^d-valued random vectors is said to be weakly associated if every finite subcollection is weakly associated.

The following theorem is an invariance principle for a strictly stationary weakly associated sequence of R^d-valued random vectors due to Burton et al.(1986). Define, for $t \in [0, 1], n \geq 1$,

$$W_n(t) = \begin{cases} n^{-1/2} \sum_{j \leq k} X_j & \text{if } t = k/n \\ \text{linear between.} \end{cases} \tag{2.1}$$

THEOREM 2.2. *(Burton et al.,1986) Let $\{X_j : j \in N\}$ be a strictly stationary weakly associated sequence of R^d-valued random vectors centered at origin and with $E||X_1||^2 < \infty$. If*

$$0 < E||X_1||^2 + 2 \sum_{j=2}^{\infty} \sum_{i=1}^{d} E(X_1^{(i)} X_j^{(i)}) = \sigma^2 < \infty, \tag{2.2}$$

then as $n \to \infty$,

$$W_n \xrightarrow{w} B^d,$$

where $\xrightarrow{w}$ *indicates weak convergence, and* B^d *is a d-dimensional Wiener process with covariance matrix* $\sum = [\sigma_{ij}]$

$$(2.3)\quad 0 < \sigma_{ij} = E(X_1^{(i)}X_1^{(j)}) + \sum_{k=2}^{\infty}(E(X_1^{(i)}X_k^{(j)}) + E(X_k^{(i)}X_1^{(j)}) < \infty.$$

We weaken the assumption of strict stationarity, and replace it by certain conditions on the moments. Using the coefficients

$$A(n) = \sup_{k\in N} \sum_{j:|j-k|\geq n} \sum_{i=1}^{d} \mathrm{Cov}(X_k^{(i)}, X_j^{(i)}), n \in N \cup \{0\},$$

and

$$A_{ij}(n) = \sup_{k\in N} \sum_{m:|m-k|\geq n} \mathrm{Cov}(X_n^{(i)}, X_m^{(j)}), n \in N \cup \{0\}, i, j = 1, \cdots, d$$

we obtain the following central limit theorem for nonstationary weakly associated random vectors.

THEOREM 2.3. *Let* $\{X_k = (X_k^{(i)}, \cdots, X_k^{(d)}) : k \in N\}$ *be a sequence of weakly associated* R^d*-valued random vectors, centered at origin with* $E||X_k||^2 < \infty$. *Assume for each* $i (i = 1, \cdots, d)$

$$(2.4)\qquad A_{ij}(n) \xrightarrow[n]{} 0 \text{ and } A_{ij}(0) < \infty$$

$$(2.5)\qquad \sup_{k\in N} E||X_k^{(i)}||^3 < \infty$$

$$(2.6)\qquad \inf_{k\in N} Var(X_k^{(i)}) > 0.$$

Then $n^{-1/2}\sum_{j=1}^{n} X_j$ *has asymptotically a normal distribution centered at origin, with covariance matrix* $\sum = [A_{ij}(0)]$.

Proof. Since $A_{ii}(n) \to 0$ and $A_{ii}(0) < \infty$ the sequence $\{X_k^{(i)} : k \in N\}$ satisfy the conditions of Cox and Grimmett(1984) central limit theorem and hence $n^{-1/2}S_n^{(i)} = n^{-1/2}\sum_{k=1}^{n} X_k^{(i)}$ is asymptotically normally distributed for each k. Finally by an application of Cramer-Wold techhnique(Theorem 7.7 of Billingsley (1968)) to this result the proof of the theorem completes.

REMARK. Note that for a wide sense stationary sequence of weakly associated random vectors (2.2) implies

$$A(0) = \sigma^2, A(n) = 2 \sum_{k=n+1}^{\infty} \sum_{i=1}^{d} \mathrm{Cov}(X_1^{(i)}, X_k^{(i)}), n \in N \cup \{0\},$$

and hence (2.4) and (2.6) are automatically statisfied. Therefore in the stationary case Theorem 2.3 is the implicit central limit theorem of Theorem 2.2 except superfluous third moment condition (2.5).

3. An invariance principle

In this section we extend Theorems 2.2 and 2.3 to an invariance principle for weakly associated random vectors (Thorem 3.1) which requires neither stationarity nor the finiteness of $A_{ij}(n)$.

THEOREM 3.1. *Let $\{X_k : k \in N\}$ be a sequence of weakly associated random vectors centered at origin and with $E||X_k||^2 < \infty$. Assume for each $i (1 \leq i \leq d)$*

$$A_{ij}(n) \xrightarrow[n]{} 0, A_{ij}(0) < \infty \tag{3.1}$$

$$\sup_{k \in N} E||X_k^{(i)}||^3 < \infty \tag{3.2}$$

$$\inf_{k \in N} Var(X_k^{(i)}) > 0 \tag{3.3}$$

and define $W_n(t)$ as in (2.1). Then as $n \to \infty$

$$W_n \xrightarrow{w} B^d$$

where $\xrightarrow{w}$ indicates weak convergence, and B^d is a d-dimensional Wiener process with covariance matrix $\sum = [A_{ij}(0)]$.

We need the following results to prove Theorem 3.1.

LEMMA 3.2. *Let $\{X_k : k \in N\}$ be a sequence of weakly associated random vectors at origin and with $E||X_k||^2 < \infty$.*
If $A_{ij}(0) < \infty$ then $\sup\{E||S_{m+n} - S_m||^2/n : m \geq 0, n \geq 1\} < \infty$.

Proof. Let $S_n^{(i)}$ denotes the ith component of S_n, that is $S_n^{(i)} = X_1^{(i)} + \cdots + X_n^{(i)}$. Then

$$\begin{aligned}
\frac{1}{n}E||S_{m+n} - S_m||^2 &= \frac{1}{n}\sum_{i=1}^{d} E(S_{m+n}^{(i)} - S_m^{(i)})^2 \\
&= \frac{1}{n}\sum_{i=1}^{d} \mathrm{Cov}(\sum_{k=m+1}^{m+n} X_k^{(i)}, \sum_{k=m+1}^{m+n} X_k^{(i)}) \\
&= \frac{1}{n}\sum_{i=1}^{d}(\sum_{j=m+1}^{m+n}\sum_{k=m+1}^{m+n} \mathrm{Cov}(X_j^{(i)}, X_k^{(i)})) \\
&\geq \sup_{k\in N} \sum_{j=|k-j|\geq 0} \sum_{i=1}^{d} \mathrm{Cov}(X_j^{(i)}, X_k^{(i)}) \\
&\qquad \text{for every } m, n (m \geq, n \geq 1) \\
&= A(0) < \infty.
\end{aligned}$$

Thus $\sup\{\frac{1}{n}E||S_{m+n} - S_m||^2 : m > 0, n \geq 0\} < \infty$.

LEMMA 3.3. *Let $\{X_k : k \in N\}$ be a sequence of weakly associated random variables with $EX_k = 0$ and $EX_k^2 < \infty$ and satisfy the central limit theorem. Assume $\sup\{E(S_{m+n} - S_m)^2/n : m \geq 0, n \geq 1\} < \infty$. Define for $t \in [0,1]$ and $n \in N$, $Y_n(t) = n^{-1/2}S_{[nt]} + (nt - [nt])(n^{-1/2}X_{[nt]+1})$. Then the sequence $\{Y_n\}$ is tight.*

Proof. Since $Y_n(0) = 0$, certainly $\{Y_n(0)\}$ is tight. Define $T_i = X_{k+1} + \cdots + X_{k+i}$. Then in the proof of Theorem 3 of Newman and Wright(1981) the line (12) with S_i replaced by $S_i = T_i$ yields that

$$P(\max(|T_1|, \cdots, |T_n|) \geq \lambda t_n) \leq 2P(|T_n| \geq (\lambda - \sqrt{2})t_n)$$

where $t_n^2 = E(T_n^2)$ and hence by taking $\lambda(E(T_n/\sqrt{n})^2)^{1/2} = \lambda'$

$$\begin{aligned}&P(\max(|T_1|,\cdots,|T_n|) \geq \lambda'\sqrt{n})\\ =&P(\max(|T_1|,\cdots,|T_n|) \geq \lambda(E(T_n/\sqrt{n})^2)^{1/2}\sqrt{n})\\ =&P(\max(|T_1|,\cdots,|T_n|) \geq \lambda t_n)\\ \leq&2P(|T_n| \geq (\lambda - \sqrt{2})t_n)\\ \leq&2P(|T_n| \geq (1/2)\lambda t_n) \text{ for } \lambda > 2\sqrt{2}.\end{aligned}$$

By the assumption of central limit theorem

$$P(|T_n| \geq 1/2\lambda t_n) \to P(|N| > (1/2)\lambda) \leq (8/\lambda^3)E\{|N|^3\}.$$

Therefore let $\epsilon > 16E(N)^3(t_n^2/n)$ then

$$P\{\max_{i\leq n}|S_{k+i} - S_k| \geq \lambda'\sqrt{n}\} \leq \epsilon/\lambda'^2 \tag{3.4}$$

holds for all k since $\sup\{E(S_{m+n} - S_m)^2/n : m \geq 0, n \geq 1\} < \infty$. Tightness of $\{Y_n\}$ now follows by Theorem 8.4 of Billingsley (1968).

LEMMA 3.4. *(Burton et al., 1986) Let $Y_1, Y_2, \cdots, Y_k$ and $Y_1', \cdots, Y_k'$ be two sets of R^d-valued random vectors having finite moment generating functions. Suppose that for all nonnegative real vectors $a_1, \cdots, a_n$ the joint distributions of $(\langle a_1, Y_1\rangle, \cdots, \langle a_k, Y_k\rangle)$ and $(\langle a_1, Y_1'\rangle, \cdots, \langle a_k, Y_k'\rangle)$ coincide. Then $(Y_1, \cdots, Y_k)$ and $(Y_1', \cdots, Y_k')$ have the same joint distribution.*

The following theorem which is an extended Newman's Inequality([10]) implies that weakly associated and uncorrelated random vectors are jointly independent.

THEOREM 3.5. *Let $X_i = (X_i^{(1)}, \cdots, X_i^{(d)}), 1 \leq i \leq N$, be a weakly associated set of R^d-valued random vectors. Let ϕ_i be the characteristic function of X_i and let ϕ^N be the joint characteristic function of $X_1, \cdots, X_N$, then for any vectors $r_1, \cdots, r_N \in R^d$ we have*

$$|\phi^N(r_1, \cdots, r_N) - \prod_{i=1}^n \phi_i(r_i)| \leq 2 \sum_{1\leq k<m\leq N} \sum_{1\leq i,j\leq d} |r_k^{(i)} r_m^{(j)}| Cov(X_k^{(i)}, X_m^{(j)}).$$

Proof of Theorem 3.1. To prove tightness of the sequence $\{W_n(\cdot), n \geq 1\}$ note that the sequence $\{X_k^{(i)} : k \in N\}$ satisfies the central limit theorem for each $i(i = 1, \cdots, d)$ from the results of Cox and Grimmett(1984) (see Theorem 2.3) and $\sup\{(S_{m+n}^{(i)} - S_m^{(i)})^2/n : m \geq 0, n \geq 1\} < \infty$ by Lemma 3.2. Tightness of the coordinate sequence $\{W_n : i = 1, \cdots, d\}$ and thereby tightness of $\{W_n(\cdot), n \geq 1\}$ itself follows by standard arguments and Lemma 3.3. Hence it remains to show that the only possible limit point is Wiener process in R^d with covariance structure $\sum = (A_{ij}(0))$. Let $Z(\cdot)$ be a limit point of $(W_n(\cdot))$. We have to prove

(i) $Z(t+h) - Z(t)$ has normal distribution with covariance $h \cdot \sum$;

(ii) Z has independent increments.

Let $a \in R^d$ be a nonnegative vector. Then $\langle a, (Z(t+h) - Z(t))\rangle$ has by Theorem 2.3 a normal distribution with variance $ha \sum a^T$. Now we can apply Lemma 3.4 with $k = 1$ which yields (i). Let $0 \leq t_1 < t_2 <, \cdots, < t_k \leq 1$ be given.

$$(W_n(t_1), W_n(t_2) - W_n(t_1), \cdots, W_n(t_k) - W_n(t_{k-1})) \underset{n}{\longrightarrow} (Z(t_1), Z(t_2) - Z(t_1), \cdots, Z(t_k) - Z(t_{k_1})) \text{ in distribution.}$$

Let $a_1, \cdots, a_k \in R^d$ be non-negative vectors. Then

$$(\langle a_1, W_n(t_1)\rangle, \cdots, \langle a_k, (W_n(t_k) - W_n(t_{k_1}))\rangle) \underset{n}{\longrightarrow} (\langle a_1, Z(t_1)\rangle, \cdots, \langle a_k, (Z(t_k) - Z(t_{k-1}))\rangle) \text{ in distribution.}$$

Since $\langle a_i, (W_n(t_i) - W_n(t_{i-1}))\rangle$ are weakly asociated by (P_4) of Esary, Proschan and Walkup[7], $\langle a_1, Z(t_1)\rangle, \cdots, \langle a_k, (Z(t_k) - Z(t_{k-1}))\rangle$ are also weakly associated according to (P_5) of Esary et al. [7].

We next consider a computation from (3.1), which yields uncorrelatedness of $\langle a_i, (Z(t_i) - Z(t_{i-1}))\rangle$. For $i \neq j$, $0 \leq t_{i-1} < t_i < t_{j-1} < t_j \leq 0$, $k = 1, \cdots, d$

$$\begin{aligned} &\mathrm{Cov}(a_i^{(k)}(W_n^{(k)}(t_i - W_n^{(k)}(t_{i-1})), a_j^{(k)}(W_n^{(k)}(t_j) - W_n^{(k)}(t_{j-1}))) \\ &= (a_i^{(k)} a_j^{(k)}/n)\mathrm{Cov}((S_{[nt_i]}^{(k)} - S_{[nt_{i-1}]}^{(k)} - S_{[nt_{j-1}]}^{(k)})) \\ &\leq (a_i^{(k)} a_j^{(k)}/n) \min([nt_i] - [nt_{i-1}], [nt_j] - [nt_{j-1}]) \\ &\quad \cdot \sup_{i \in N} \sum_{j:|j-i| \geq [nt_{j-1}] - [nt_i]} \mathrm{Cov}(X_i^{(k)}, X_j^{(k)}) \underset{n}{\longrightarrow} 0 \end{aligned} \tag{3.5}$$

and for $r \neq s, i \neq j$

$$
\begin{aligned}
&\mathrm{Cov}(a_i^{(r)}(W_n(t_i)^{(r)} - W_n(t_{i-1})^{(r)}), a_j^{(s)}(W_n(T_j)^{(s)} - W_n(t_{j-1}^{(s)})) \\
&= a_i^{(r)} a_j^{(s)}/n \mathrm{Cov}((S_{[nti]}^{(r)} - S_{[nti-1]}^{(r)}), (S_{[ntj]}^{(s)} - S_{[ntj-1]}^{(s)})) \\
&\leq a_r^{(r)} a_j^{(s)}/n \min(([nt_i] - [nt_{i-1}]), ([nt_j] - [nt_{j-1}])) \\
&\qquad \cdot \sup_{i \in N} \sum_{j:|j-1| \geq [ntj-1]-[nti]} \mathrm{Cov}(X_i^{(r)}, X_j^{(s)}) \underset{n}{\longrightarrow} 0.
\end{aligned} \tag{3.6}
$$

(3.5) and (3.6) yield for $i \neq j, 0 \leq t_{i-1} < t_i < t_{j-1} < t_j \leq 1$

$$
\begin{aligned}
&\mathrm{Cov}(\langle a_i, (Z(t_i) - Z(t_{i-1}))\rangle, \langle a_j, (Z(t_j) - Z(t_{j-1}))\rangle) \\
&= \lim_{n \to \infty} (\langle a_i, (W_n(t_i) - W_n(t_{i-1}))\rangle, \langle a_j, (W_n(t_j) - W_n(t_{j-1}))\rangle) = 0.
\end{aligned}
$$

Hence the $\langle a_i, (Z(t_i) - Z(t_{i-1}))\rangle$ are associated and uncorrelated, which together imply independence by Theorem 3.5. Now we again apply Lemma 3.4 to obtain the independence of the increments of the Z process.

The following lemma proved by ideas of Newman([9,10]) will be used to prove Theorem 3.5.

LEMMA 3.6. *If* (X_1, Y_1) *and* (X_2, Y_2) *are weakly associated. Then*

$$|Cov(\exp i(X_1 - Y_1), \exp i(X_2 - Y_2))| \leq 2 Cov(X_1 + Y_1, X_2 + Y_2).$$

Proof. Since for real f and G

$$
\begin{aligned}
|\mathrm{Cov}(f, g)| &= \sup_{\alpha \in R}(Re(\exp i\alpha Cov(f, g) = \sup_{\alpha \in R}(Re\mathrm{Cov}(f, g \exp i\alpha) \\
&= \sup_{\alpha \in R} \mathrm{Cov}(f, Re(g \exp i\alpha),
\end{aligned}
$$

we have

$$
\begin{aligned}
&|\mathrm{Cov}(\cos(X_1 - Y_1), \exp i(X_2 - Y_2))| \\
&\qquad = \sup_{\alpha} \mathrm{Cov}(\cos(X_1 - Y_1), Re(\exp i(X_2 - Y_2 + \alpha)))
\end{aligned} \tag{3.7}
$$

and

$$\text{(3.8)}\qquad \begin{aligned}&|\mathrm{Cov}(\sin(X_1 - Y_1), \exp i(X_2 - Y_2))|\\ &\quad = \sup_{\beta} \mathrm{Cov}(\sin(X_1 - Y_1), Re(\exp i(X_2 - Y_2 + \beta)))\end{aligned}$$

which yield

$$\text{(3.9)}\qquad \begin{aligned}&|\mathrm{Cov}(\exp i(X_1 - Y_1), \exp i(X_2 - Y_2))|\\ &\quad \leq |\mathrm{Cov}(\cos(X_1 - Y_1), \exp i(X_2 - Y_2))|\\ &\quad + |\mathrm{Cov}(\sin(X_1 - Y_1), \exp i(X_2 - Y_2))|\\ &\quad = \sup_{\alpha} \mathrm{Cov}(\cos(X_1 - Y_1), Re(\exp i(X_2 - Y_2 + \alpha)))\\ &\quad + \sup_{\beta} \mathrm{Cov}(\sin(X_1 - Y_1), Re(\exp i(X_2 - Y_2 + \beta))).\end{aligned}$$

To show each term inside the supremum of (3.9) bounded by $\mathrm{Cov}(X_1 + Y_1, X_2 + Y_2)$, we write

(3.10)

$$\begin{aligned}&\mathrm{Cov}(X_1 + Y_1, X_2 + Y_2) - \mathrm{Cov}(\cos(X_1 - Y_1), Re(\exp i(X_2 - Y_2 + \alpha)))\\ &= 1/2[\mathrm{Cov}(X_1 + Y_1 - \cos(X_1 - Y_1), X_2 + Y_2 + Re(\exp i(X_2 - Y_2 + \alpha)))\\ &+ \mathrm{Cov}(X_1 + Y + 1 + \cos(X_1 - Y_1), X_2 + Y_2 - Re(\exp i(X_2 - Y_2 + \alpha)))].\end{aligned}$$

(3.11)

$$\begin{aligned}&\mathrm{Cov}(X_1 + Y_1, X_2 + Y_2) - \mathrm{Cov}(\sin(X_1 - Y_1), Re(\exp i(X_2 - Y_2 + \beta)))\\ &= 1/2[\mathrm{Cov}(X_1 + Y_1 - \sin(X_1 - Y_1), X_2 + Y_2 + Re(\exp i(X_2 - Y_2 + \beta)))\\ &+ \mathrm{Cov}(X_1 + Y_1 - \sin(X_1 - Y_1), X_2 + Y_2 - Re(\exp i(X_2 - Y_2 + \beta)))].\end{aligned}$$

Since the partial derivatives of all four functions of (3.10) and (3.11) are nonnegative they are coordinatewise incresing with respect to $X_1, Y_1 (X_2, Y_2$, respectively) and hence the covariances are nonnegative by weak association. This completes the proof of the lemma.

Proof of Theorem 3.5. Let

$$V_1 = \sum_{k=1}^{N-1} \sum_{i=1}^{d} r_{ki} + X_k^{(i)}, V_2 = \sum_{i=1}^{d} r_{Ni} + X_N^{(i)},$$

and

$$W_1 = \sum_{k=1}^{N-1} \sum_{i=1}^{d} r_{ki} - X_k^{(i)}, W_2 = \sum_{i=1}^{d} r_{Ni} - X_N^{(i)},$$

where for any real s we defines $s^+ = \max(s, 0)$ and $s^- = \max(-s, 0)$. Since (V_1, W_1) and (V_2, W_2) are weakly associated by the triangle inequality and Lemma 3.6 we obtain

$$\begin{aligned}
&|\Phi^N(r, \cdots, r_N) - \prod_{i=1}^{N} \Phi_i(r_i)| \\
&\leq \Phi^N(r_1, \cdots, r_N) - \Phi^{N-1}(r_1, \cdots, r_{N-1})\Phi_N(r_N)| \\
&\qquad + |\Phi^{N-1}(r_1, \cdots, r_{N-1})\Phi_N(r_N) - \prod_{i=1}^{N-1} \Phi_i(r_i)| \\
&\leq |\mathrm{Cov}(\exp i(V_1 - W_1), \exp i(V_2 - W_2))| \\
&\qquad + |\Phi^{N-1}(r_1, \cdots, r_{N-1}) - \prod_{i=1}^{N-1} \Phi_i(r_i)| \\
&\leq 2\mathrm{Cov}(V_1 + W_1, V_2 + W_2) + |\Phi^{N-1}(r_1, \cdots, r_{N-1}) - \prod_{i=1}^{N} \Phi(r_i)| \\
&\leq 2 \sum_{k=1}^{N-1} \sum_{i=1}^{d} \sum_{j=1}^{d} |r_{ki} r_{Nj}| \mathrm{Cov}(X_{ki}, \cdots, X_{Nj}) \\
&\qquad + |\Phi^{N-1}(r_1, \cdots, r_{N_1}) - \prod_{i=1}^{N-1} \Phi_i(r_i)|.
\end{aligned}$$

The desired result follows by induction.

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Department of Statistics
Wonkwang University
Iri 570-749, Korea
and
Department of Computer Science
Hyejeon Junior College
Hongsung 350-800, Korea