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SEQUENTIAL CONVERGENCE SPACES

WOO CHORL HONG

1. Introduction

Since the early 1960's many topologists have characterized the class of topological spaces which can be specified completely by the knowledge of their convergent sequences. It is a well known and useful fact that every first-countable space falls into this class.

A. Arhangel'skiǐ introduced a Fréchet space [1], which satisfies the following property (F):

(F) The closure of any subset A of a topological space X is the set of limits of sequences in A.

Clearly, each first-countable space (and so each metric space) is a Fréchet space. In [2] and [3], S.P. Franklin introduced a concept of sequential spaces which is also a generalization of first-countable spaces and he investigated properties of Fréchet spaces and relations between sequential spaces and Fréchet spaces. And related concepts have been studied in detail by many authors (See [4], [5] and [6]).

In this paper, we shall introduce and study sequential convergence structures and show that Fréchet spaces are determined completely by these structures.

2. Sequential convergence structures and Fréchet spaces

Let X be a non-empty set and let $S(X)$ be the set of all sequences in X. Sequences in X will be denoted by small Greek letters α, β etc. The k-th term of the sequence α is denoted by $\alpha(k)$. The small Latin letters s, t denote monotone increasing mappings of the natural number set N into itself. The composition $\alpha \circ s$ is the subsequence of α which has $\alpha(s(k))$ as k-th term.

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A non-empty subfamily L of $S(X) \times X$ is called a sequential convergence structure on X if it satisfies the following properties;

(SC1) For each $x \in X, ((x), x) \in L$, where (x) is the constant sequence whose k-th term is x for all indices k.

(SC2) If $(a, x) \in L$, then $(\alpha \circ s, x) \in L$ for each s.

(SC3) Let $x \in X$ and $A \subset X$. If $(\alpha, x) \notin L$ for each α in A, then $(\beta, x) \notin L$ for each β in $\{y \in X | (\alpha, y) \in L$ for some α in $A\}$.

If a sequential convergence structure L on X is given, the pair (X, L) is called a sequential convergence space. Hereafter we use the notation $SC[X]$ for the set of all sequential convergence structures on X.

THEOREM 1. *For* $L \in SC[X]$, *define a mapping* $C_L : P(X) \longrightarrow P(X)$ *as follows : for each subset* A *of* X,

$$C_L(A) = \{x \in X | (\alpha, x) \in L \text{ for some } \alpha \text{ in } A\}.$$

Then, C_L *is a Kuratowski closure operator on* X, *that is,* (X, C_L) *is a topological space.*

Proof. It is clear that (1) $C_L(\emptyset) = \emptyset$ and (2) for each subset A of X, $A \subset C_L(A)$ by (SC1).

(3) Let A be a non-empty subset of X. Note that (SC3) means $C_L(C_L(A)) \subset C_L(A)$.

(4) Let A and B be non-empty subsets of X. By (2), we have $C_L(A) \cup C_L(B) \subset C_L(A \cup B)$. It is enough to show that $C_L(A \cup B) \subset C_L(A) \cup C_L(B)$. Let $x \in C_L(A \cup B)$. Then $(\alpha, x) \in L$ for some α in $A \cup B$. Note that either A or B contains infinitely many terms of α. If A countains infinitely many terms of α, then there exists a subsequence $\alpha \circ s$ of α in A with $(\alpha \circ s, x) \in L$ by (SC2).

Let $\mathcal{L}(C_L)$ denote the set of all pairs (α, x) such that α converges to x in the topological space (X, C_L). Now we are going to determine the relations between L and $\mathcal{L}(C_L)$. First of all we shall prove the following

LEMMA 2. *Let* $L \in SC[X]$ *and* $x \in A \subset X$. *Then,* A *is a nbd of* x *in* (X, C_L) *iff for each* $(\alpha, x) \in L, \alpha$ *is eventually in* A.

Proof. Let A be a nbd of x in (X, C_L) and $(\alpha, x) \in L$. Since A is a nbd of x in (X, C_L), there exists an open set O in (X, C_L) such that

$x \in O \subset A$. It follows that $C_L(X \setminus O) = X \setminus O$, and thus there does not exist β in $X \setminus O$ such that $(\beta, x) \in L$ by definition of C_L. We now prove that $\{k \in N | \alpha(k) \in X \setminus O\}$ is finite. If $\{k \in N | \alpha(k) \in X \setminus O\}$ is infinite, then there exists a subsequence $\alpha \circ s$ of α in $X \setminus O$. Since $(\alpha, x) \in L, (\alpha \circ s, x) \in L$ by (SC2), which is a contradiction. From this fact, we have α is eventually in O. Therefore, α is enentually in A.

Conversely, suppose that A is not a nbd of x in (X, C_L). Then $x \in C_L(X \setminus A)$. By definition of C_L, we have a sequence α in $X \setminus A$ such that $(\alpha, x) \in L$. We note that α is not eventually in A.

THEOREM 3. *Let $L \in SC[X]$. Then, we have*
(1) $L \subset \mathcal{L}(C_L)$ and
(2) $C_L = C_{\mathcal{L}(c_L)}$.

Proof. (1) Let $(\alpha, x) \in L$. Then, by Lemma 2, for each nbd A of x in $(X, C_L), \alpha$ is eventually in A. Hence α converges to x in (X, C_L), and therefore $(\alpha, x) \in \mathcal{L}(C_L)$.

(2) Let A be a non-empty subset of X. Then, by (1), we have $C_L(A) \subset C_{\mathcal{L}(C_L)}(A)$. Conversely, let $x \in C_{\mathcal{L}(C_L)}(A)$. Then $(\alpha, x) \in \mathcal{L}(C_L)$ for some α in A. By definition of $\mathcal{L}(C_L)$, α converges to x in (X, C_L) and so $x \in C_L(A)$.

COROLLAPY 4. *Let $L \in SC[X]$. Then we have*
(1) $\mathcal{L}(C_L) \in SC[X]$ and
(2) $\cup\{L' \in SC[X] | C_L = C_{L'}\} = \mathcal{L}(C_L)$.

Proof. (1) The set $\mathcal{L}(C_L)$ satisfies (SC1) and (SC2) obviously. By Theorem 3 (2), $\mathcal{L}(C_L)$ satisfies (SC3).

(2) It is immediate from (1) and Theorem 3 (2).

EXAMPLE 5. In general, $L \neq \mathcal{L}(C_L)$. Let Q be the rational number set with usual topology. Let L_Q denote the set of all pairs $(\alpha, x) \in S(Q) \times Q$ such that α converges to x in Q and $L = \{((x), x) | x \in Q\} \cup \{(\alpha, x) \in S(Q) \times Q | \alpha$ converges to x in Q and α is either increasing or decreasing$\}$. Then $L_Q, L \in SC[Q]$. Since C_{L_Q} is the closure operator in the usual space Q, $\mathcal{L}(C_{L_Q}) = L_Q$. Moreover, it is easy to see that $C_{L_Q} = C_L$. Hence $L \subsetneq L_Q = \mathcal{L}(C_{L_Q}) = \mathcal{L}(C_L)$.

Finally, we shall investigate relations between sequential convergence structures and Fréchet topologies (spaces).

THEOREM 6. *(1) For each $L \in SC[X], (X, C_L)$ is a Fréchet space.*

(2) For each Fréchet topology $\Im$ on X, $L_\Im = \mathcal{L}(C_{L_\Im}) \in SC[X]$, where $L_\Im = \{(\alpha, x) \in S(X) \times X | \alpha$ converges to x in $(X, \Im)\}$.

Proof. (1) It is immediate from definition of C_L and Theorem 3 (1).

(2) Note that $C_{L_\Im}$ is the closure operator for $(X, \Im)$, since $\Im$ is a Fréchet topology. Hence $L_\Im = \mathcal{L}(C_{L_\Im})$.

COROLLARY 7. *Let $F[X]$ denote the set of all Fréchet topologies on X and let $TSC[X] = \{\mathcal{L}(C_L) | L \in SC[X]\}$. Then, partially ordered sets $F[X]$ and $TSC[X]$ endowed with the set inclusion order are dual-isomorphic under the correspondence $\Im \longrightarrow L_\Im$.*

Proof. Sich $C_{L_\Im}$ is the closure operator for $(X, \Im)$, $L_{\Im_1} = L_{\Im_2}$ implies $\Im_1 = \Im_2$. Hence this correspondence is one-to-one. Take any L in $SC[X]$ and let $\Im_{C_L}$ be the Fréchet topology on X with the closure operator C_L. Then $L_{\Im_{C_L}} = \mathcal{L}(C_L)$. Thus this correspondence is onto.

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Department of Mathematics
Pusan National University
Pusan 609-735, Korea