Comm. Korean Math. Soc. 7 (1992), No. 2, pp. 189–207

INTEGRODIFFERENTIAL EQUATIONS WITH TIME DELAY IN HILBERT SPACE

DONG GUN PARK AND SEON YU KIM

1. Introduction

In this paper we consider two types of integro-differential equations containing time delay in a complex Hilbert space H. One is an equation of Volterra type

$$(1.1)\quad \frac{d}{dt}u(t) + Au(t) + \int_0^t a(t-s)Bu(s)ds = f(t), \quad 0 < t < T,$$

$$(1.2)\quad u(0) = x.$$

The other is the following retarded functional differential equation

$$(1.3)\quad \frac{d}{dt}u(t) + Au(t) + A_1u(t-h) + \int_{-h}^0 a(-s)A_2u(t+s)ds = f(t), \quad 0 < t < T,$$

$$(1.4)\quad u(0) = x, \ u(s) = y(s), \quad s \in [-h, 0).$$

Here, A is the operator associated with a sesquilinear form $a(u,v)$ defined in $V \times V$ and satisfying Garding's inequality and (2.2) of Section 2 where V is another Hilbert space such that $V \subset H \subset V^*$. B, A_1, A_2 are bounded linear operators from V to V^* such that BA^{-1}, A_1A^{-1}, A_2A^{-1} map H into itself boundedly. The function a in (1.1), is a numerical valued function of bounded variation in the interval $[0,T]$, and that in (1.3) is a similar function in $[-h, 0]$. The right member f is some function with values in H.

We try to solve (1.1) and (1.2) for an arbitrary $x \in H$ and f which does not necessarily belong to $L^1(0,T;H)$, assuming some other conditions

Received April 29, 1991.
This research is supported by the Ministry of Education, Korea in 1990.

(F.i) and (F.ii) of Section 2 instead. An example of such a function $f \notin L^1(0,T;H)$ is given in the appendix.

Following the method in M. G. Crandall and J. A. Nohel [5] we can reduce (1.1) to the equation

$$\frac{du}{dt} + Au = G(u) \tag{1.5}$$

where letting R be the solution of

$$aBA^{-1} + R + aBA^{-1} * R = 0$$

$G(u)$ is defined by

$$G(u) = f + R * f - R(0)u + Rx - \dot{R} * u. \tag{1.6}$$

The function R is of bounded variation with values in $B(H)$ as well as in $B(V^*)$, and $G(u)$ will be considered as a function with values in H and also in V^* for $u \in C([0,T];H)$. Since f is not assumed to belong to $L^1(0,T;H)$, $R * f$ in the right side of (1.6) is defined as an improper integral

$$(R * f)(t) = \int_{+0}^{t} R(t-s)f(s)ds = \lim_{\varepsilon \to +0} \int_{\varepsilon}^{t} R(t-s)f(s)ds,$$

when it is considered as a function taking values in H. Then (1.5) and (1.2) can be solved by successive approximation, which establish the existence and uniqueness of a solution u of (1.1) and (1.2) such that

$$u \in L^2(0,T;V) \cap C([0,T];H)$$
$$u', Au \in L^2(0,T;H,tdt).$$

The third term of the left side of (1.1) exists as a Bocher integral in V^*, but it should be understood in the improper sense when it is considered as an integral in H:

$$\int_{+0}^{t} a(t-s)Bu(s)ds = \lim_{\varepsilon \to +0} \int_{\varepsilon}^{t} a(t-s)Bu(s)ds.$$

Next, we apply this result to the second equation (1.3) under the same assumption for f as above. Assuming

$$x \in H, \ y \in L^2(-h,0;V) \cap L^2(-h,0;D(A),(s+h)ds)$$

$$\int_{-h+0}^{0} Ay(s)ds \in H,$$

we can solve (1.3) and (1.4) step by step in $[nh,(n+1)h \wedge T]$ for $n = 0, 1, \ldots,$ and show the existence and uniqueness of the solution u such that

$$u \in C([nh,(n+1)h \wedge T]; H) \cap L^2(nh,(n+1)h \wedge T; V),$$

$$u', \ Au \in L^2(nh,(n+1)h \wedge T; H,(t-nh)dt),$$

$$\int_{nh+0}^{(n+1)h \wedge T} Au(t)dt \in H,$$

for any integer n such that $nh < T$. The integral in the left side of (1.3) should be understood in the improper sense

$$\{\int_{-h}^{nh-t} + \int_{nh-t+0}^{0}\} a(-s)A_2 u(t+s)ds$$

when it is considered as an integral in H.

The author wishes to express the deepest appreciation to Professor H. Tanabe of Osaka University for his valuable suggestions and encouragement.

2. Assumption and main theorems

Let H and V be complex Hilbert spaces such that V is a dense subspace of H and the embedding of V to H is continuous. The norms of H and V are denoted by $|\ |$ and $||\ ||$, respectively. Identifying H with its antidual we may consider $V \subset H \subset V^*$. The norm of V^* is denoted by $||\ ||_*$. For a couple of Hilbert spaces X and Y the notation

$B(X,Y)$ denotes the totality of bounded linear operators from X to Y, and $B(X) = B(X,X)$.

Let $a(u,v)$ be a sesquilinear form defined in $V \times V$. Suppose that there exists positive constants C_1, c, c_1 such that

$$|a(u,v)| \le C_1 \|u\| \, \|v\| \tag{2.1}$$

$$\mathrm{Re}\, a(u,u) \ge c\|u\|^2 - c_1|u|^2 \tag{2.2}$$

for any $u, \ v \in V$.

Let A be the operator associated with the sesquilinear form $a(u,v)$:

$$a(u,v) = (Au, v), \qquad u, v \in V. \tag{2.3}$$

The operator A belongs to $B(V, V^*)$. The realization of A in H which is the restriction of A to

$$D(A) = \{u \in V : Au \in H\}$$

is also denoted by the same letter A. It is known that $-A$ generates an analytic semigroup in both of V^* and H.

We assume that there exist a positive constant C_2 such that

$$|a(u,v) - a^*(u,v)| \le C_2\|u\||v|. \tag{2.4}$$

The operator A^* associated with $a^*(u,v)$ is the adjoint of A. From (2.4) it follows that $(A - A^*)u \in H$ for any $u \in V$ and

$$|(A - A^*)u| \le C_2\|u\|. \tag{2.5}$$

Let B, A_1, A_2 be operators belonging to $B(V, V^*)$. We assume that their restrictions to $D(A)$ all belong to $B(D(A), H)$, where $D(A)$ is a Hilbert space with the graph norm of A.

As for the inhomogeneous term f we assume

$$f \in L^2(0,T;V^*) \cap L^2(0,T;H,tdt) \tag{F.i}$$

where $f \in L^2(0,T;H,tdt)$ means that f is a strongly measurable function with values in H in $(0,T)$ and $\int_0^T |f(t)|^2 t dt < \infty$,

(F.ii) $\int_{+0}^T f(t)dt = \lim_{\varepsilon \to +0} \int_\varepsilon^T f(t)dt$ exists in H.

Concerning the function a and the initial value x we assume that for the problem (1.1), (1.2)

(I.i) a is a complex valued function of bounded variation in [0,T],

(I.ii) $x \in H$;

and for the problem (1.3), (1.4)

(II.i) h is some fixed positive number,

(II.ii) a is a complex valued function of bounded varivation in $[0,h]$,

(II.iii) $x \in H$,

(II.iv) $y \in L^2(-h,0;V) \cap L^2(-h,0;D(A),(s+h)ds)$
where $y \in L^2(-h,0;D(A),(s+h)ds)$ means that y is a strongly measurable function with values in $D(A)$ in (-h,0) and

$$\int_{-h}^0 (|Ay(s)|^2 + |y(s)|^2)(s+h)ds < \infty,$$

(II.v) $\int_{-h+0}^0 Ay(s)ds = \lim_{\varepsilon \to +0} \int_{-h+\varepsilon}^0 Ay(s)ds$ exists in H.

DEFINITION 2.1. A strong solution u of (1.1), (1.2) is a function $u \in L^2(0,T;V) \cap C([0,T];H)$ such that $u(0) = x$, u is absolutely continuous as a function taking values in H in $[\delta,T]$ for any $\delta > 0$, $u(t) \in D(A)$ a.e in $[0,T]$ and $Au \in L^2(\delta,T;H)$ for any $\delta > 0$, the improper integral

$$\int_{+0}^t a(t-s)Bu(s)ds = \lim_{\varepsilon \to +0} \int_\varepsilon^t a(t-s)Bu(s)ds \tag{2.6}$$

exists in H for any $t \in (0,T)$, and (1.1) holds a.e in $[0,T]$, where the integral in the left side of (1.1) is understood as the improper integral (2.6).

DEFINITION 2.2. A strong solution u of (1.3), (1.4) is a function $u \in L^2(-h,T;V) \cap C([0,T];H)$ such that $u(0) = x$, $u(s) = y(s)$ a.e. in $[-h,0)$, u is absolutely continuous in $[nh+\delta,(n+1)h \wedge T]$ for each $n = 0, 1, \ldots, [T/h]$ and $\delta > 0$ as a function with values in H, $u(t) \in D(A)$

a.e. in $[0,T]$ and $Au \in L^2(nh+\delta,(n+1)h \wedge T; H)$ for $n = 0,1,\dots,[T/h]$ and $\delta > 0$, the improper integral

(2.7)
$$\int_{-h}^{0} a(-s)A_2u(t+s)ds = \lim_{\varepsilon \to +0} \Big(\int_{-h}^{nh-t} + \int_{nh-t+\varepsilon}^{0} \Big) a(-s)A_2u(t+s)ds$$

exists in H for $t \in (nh,(n+1)h \wedge T)$, $n = 0, 1, \dots, [T/h]$, and (1.3) holds a.e. in $[0,T]$, where the integral in the left side of (1.3) is understood as the improper integral (2.7).

THEOREM 1. *A strong solution u of (1.1), (1.2) exists and is unique, and we have $u', Au \in L^2(0,T; h, tdt)$.*

THEOREM 2. *A strong solution u of (1.3) and (1.4) exists and is unique, and we have*

$$u', Au \in L^2(nh,(n+1)h \wedge T; H,(t-nh)dt)$$

for any nonnegative integer n such that $nh < T$.

3. Proof of Theorem 1

Since the solution u we are seeking belongs to $L^2(0,T;V) \cap C([0,T];H)$ and $f \in L^2(0,T;V^*)$ by the assumption (F.i), the function $a * Bu$ and $G(u)$ both belong to $L^2(0,T;V^*)$. Hence, u', $Au \in L^2(0,T;V^*)$ in view of J. L. Lions ([7]:Theorem 1.1). Thus, in the proof of the equivalence of (1.1) and (1.5) we can argue in the space V^* so that all integrals which appear are ones in Bocher's sense. Hence,

$$(R * f)(t) = \int_{+0}^{t} R(t-s)f(s)ds$$

exists as an improper integral in H, and is bounded:

$$|(R * f)(t)| \le |R(0)| |\int_{+0}^{t} f(\sigma)d\sigma| + V(R;[0,t]) \sup_{0 \le s \le t} |\int_{+0}^{s} f(\sigma)d\sigma|, \tag{3.1}$$

LEMMA 3.1. *Let u be the solution of*

$$u'(t) + Au(t) = g(t), \qquad 0 < t \le T, \tag{3.2}$$

$$u(0) = x, \tag{3.3}$$

where $x \in H$ and $g \in L^2(0,T;V^*)$. Then

$$|u(t)|^2 + c\int_0^t \|u(s)\|^2 ds \le |x|^2 + \frac{1}{c}\int_0^t \|g(s)\|_*^2 ds. \tag{3.4}$$

If $g \in L^2(0,T;H,tdt)$ in addition, then

$$\begin{aligned}\int_0^t |u'(t)|^2 s ds \le (1 + \frac{C_2'^2}{2c}t)|x|^2 \\ + \frac{1}{c}(1+\frac{C_2^2}{2c}t)\int_0^t \|g(s)\|_*^2 ds + 2\int_0^t |g(s)|^2 s ds.\end{aligned} \tag{3.5}$$

Proof. The inequality (3.4) is well known (see J. L. Lions [7], Theorem 1.1). The second inequality (3.5) is also rather well known, and we only sketch the proof. Assuming that $u(t)$ is a nice function we make formal calculations which are easily justified by approximating x and $f(t)$ by sequences of nice elements. Taking inner product of both sides of (3.2) and $u(t)$, and using (2.2) ($c_1 = 0$), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \mathrm{Re}a(u(t),u(t)) = \mathrm{Re}(g(t),u(t)) \\ \le &\frac{1}{2c}\|g(t)\|_*^2 + \frac{c}{2}\|u(t)\|^2 \le \frac{1}{2c}\|g(t)\|_*^2 + \frac{1}{2}\mathrm{Re}a(u(t),u(t)),\end{aligned}$$

from which we readily obtain

$$\frac{d}{dt}|u(t)|^2 + \mathrm{Re}a(u(t),u(t)) \le \frac{1}{c}\|g(t)\|_*^2.$$

Integrating this inequality from 0 to t, we have

$$|u(t)|^2 + \int_0^t \mathrm{Re}a(u(s),u(s))ds \le |x|^2 + \frac{1}{c}\int_0^t \|g(s)\|_*^2 ds, \tag{3.6}$$

and we also get

$$|u'(t)|^2 + \frac{1}{2}\frac{d}{dt}a(u(t), u(t))$$
$$= \mathrm{Re}(g(t), u'(t)) + \frac{1}{2}(u'(t), (A^* - A)u(t)).$$

Multiplying both sides of this equality by t and integrating from 0 to t we get after an integration by parts

$$\int_0^t |u'(s)|^2 s ds + \frac{t}{2}a(u(t), u(t))$$
$$= \frac{1}{2}\int_0^t a(u(s), u(s))ds + \mathrm{Re}\int_0^t (g(s), u'(s))sds$$
$$+ \frac{1}{2}\int_0^t (u'(s), (A^* - A)u(s))sds.$$

We obtain

$$\text{(3.7)} \qquad \frac{1}{2}\int_0^t |u'(s)|^2 s ds$$
$$\leq \frac{1}{2}(1 + \frac{1}{2c}C_2^2 t)\int_0^t \mathrm{Re}a(u(s), u(s))ds + \int_0^t |g(s)|^2 sds.$$

From (3.6) and (3.7) we conclude

$$\frac{1}{2}\int_0^t |u'(s)|^2 s ds$$
$$\leq \frac{1}{2}(1 + \frac{1}{2c}C_2^2 t)(|x|^2 + \frac{1}{c}\int_0^t \|g(s)\|_*^2 ds) + \int_0^t |g(s)|^2 sds$$

which completes the proof.

Set $u_0(t) \equiv x$. Let u_1 be the solution of the following initial value problem

$$\frac{d}{dt}u(t) + Au_1(t) = G(u_0)(t),$$
$$u_1(0) = x.$$

Since $u_0 \in C([0,T] : H)$ and G maps $C([0,T];H)$ to $L^2(0,T;V^*)$, $G(u_0) \in L^2(0,T : V^*)$. Hence, by the above mentioned result of J.L. Lions [7], the solution $u_1(t)$ exists.

Since $u_1(t) \in C([0,T];H)$, $G(u_1) \in L^2(0,T;V^*)$. Hence, we can define $u_2(t)$ as the solution of

$$\frac{d}{dt}u_2(t) + Au_2(t) = G(u_1)(t),$$
$$u_2(0) = x.$$

Iterating this process, we can show that there exist a sequence $\{u_n(t)\}$ such that

$$\frac{d}{dt}u_n(t) + Au_n(t) = G(u_{n-1})(t),$$
$$u_n(0) = x$$

for any $n = 1, 2, \ldots$.

To prove the convergence of $\{u_n(t)\}$, we prove the following lemma.

LEMMA 3.2. *Let u and $\widehat{u}$ be elements of $C([0,T];H)$, and $v(t)$ and $\widehat{v}(t)$ be solutions of the following equations :*

$$\frac{d}{dt}v(t) + Av(t) = G(u)(t), \quad v(0) = x,$$
$$\frac{d}{dt}\widehat{v}(t) + A\widehat{v}(t) = G(\widehat{u})(t), \quad \widehat{v}(0) = x.$$

Then the following inequality holds :

$$(3.8) \qquad |v(t) - \widehat{v}(t)| \leq (|R(0)| + V(R;0,t)) \int_0^t |u(s) - \widehat{u}(s)| ds.$$

Proof. Taking the inner product of both sides of

$$\frac{d}{dt}(v(t) - \widehat{v}(t)) + A(v(t) - \widehat{v}(t)) = G(u)(t) - G(\widehat{u})(t)$$

and $(v(t) - \widehat{v}(t))$, we obtain

$$\frac{1}{2}\frac{d}{dt}|v(t) - \widehat{v}(t)|^2 \leq |G(u)(t) - G(\widehat{u})(t)|\,|v(t) - \widehat{v}(t)|.$$

By Lemma A.5 of page 157 of [4], we have

$$|v(t) - \widehat{v}(t)| \leq \int_0^t |G(u)(s) - G(\widehat{u})(s)|ds.$$

Note that $G(u)$ and $G(\widehat{u})$ themselves do not belong to $L^1(0,T;H)$, but their difference does. By the definition of $G(u)(t)$, we obtain

$$G(u)(s) - G(\widehat{u})(s) = -R(0)(u(s) - \widehat{u}(s)) - (\dot{R} * (u - \widehat{u}))(s).$$

Hence

$$|v(t) - \widehat{v}(t)| \leq R(0) \int_0^t |u(s) - \widehat{u}(s)|ds + \int_0^t |(\dot{R} * (u - \widehat{u}))(s)|ds.$$

By an elementary calculation, we obtain (3.8).

Applying (3.8) to u_n, u_{n-1} in place of u, $\widehat{u}$

$$|u_{n+1}(t) - u_n(t)| \leq (|R(0)| + V(R;0,t)) \int_0^t |u_n(s) - u_{n-1}(s)|ds.$$

If $0 \leq t \leq T$ then $V(R;0,t) \leq V(R;0,T)$. Hence, putting

$$C_0 = |R(0)| + V(R;0,T),$$

we have

$$|u_{n+1}(t) - u_n(t)| \leq C_0 \int_0^t |u_n(s) - u_{n-1}(s)|ds. \tag{3.9}$$

Iterating (3.9) one shows by induction that

$$|u_{n+1}(t) - u_n(t)| \leq \frac{(C_0T)^n}{n!} \max_{0 \leq \tau \leq T} |u_1(\tau) - u_0(\tau)|,$$

which implies that $\{u_n(t)\}$ converges uniformly in H.

Put $u(t) = \lim_{n\to\infty} u_n(t)$. Applying (3.4), (3.5) to u_{n+1}, we get

$$c\int_0^t \|u_{n+1}(s)\|^2 ds \le |x|^2 + \frac{1}{c}\int_0^t \|G(u_n)(s)\|_*^2 ds,$$

$$\int_0^t |u'_{n+1}(s)|^2 s ds \le (1+\frac{C^2}{2c}t)|x|^2 + \frac{1}{c}(1+\frac{C^2}{2c}t)\int_0^t \|G(u_n)(s)\|_*^2 ds$$
$$+ 2\int_0^t |G(u_n)(s)|^2 s ds.$$

As is easily seen the right hand sides of the above inequalities are uniformly bounded. Hence, we see that u and u' belong to $L^2(0,T:V)$ and $L^2(0,T:H,t\,dt)$, respectively, and u satisfies (1.5) and (1.2). Thus u is a solution of (1.1) and (1.2).

For $0<\varepsilon<t$

$$\int_\varepsilon^t a(t-s)Bu(s)ds = \int_\varepsilon^t a(t-s)BA^{-1}(G(u)(s)-u'(s))ds \tag{3.10}$$

$$= \int_\varepsilon^t a(t-s)BA^{-1}(G(u)(s)-f(s))ds + BA^{-1}\int_\varepsilon^t a(t-s)(f(s)-u'(s))ds.$$

Since $G(u)(s)-f(s)$ is bounded in H in view of (3.1), the first term in the right side of (3.10) converges in H as $\varepsilon\to+0$. Since

$$\int_\varepsilon^t a(t-s)(f(s)-u'(s))ds$$
$$= \int_\varepsilon^t a(t-s)\frac{d}{ds}(\int_{+0}^s f(\sigma)d\sigma - u(s))ds$$
$$= a(0)(\int_{+0}^t f(\sigma)d\sigma - u(t)) - \int_{+0}^t d_s a(t-s)(\int_{+0}^s f(\sigma)d\sigma - u(s)) + a(t)x$$

as $\varepsilon\to+0$, the second term also converges in H as $\varepsilon\to+0$. Hence,

$$\int_{+0}^t a(t-s)Bu(s)ds$$

exists as an improper integral in H for each $t\in(0,T]$.

Uniqueness of solutions of (1.1) and (1.2) follows from Lemma 3.2.

4. Proof of Theorem 2

We have only to consider the case $h < T$. First we show the existence of solutions of (1.3) and (1.4) in $[0,h]$. For $0 < t < h$ the problems (1.3) and (1.4) are reduced to

$$(4.1)\quad \frac{d}{dt}u(t) + Au(t) + \int_0^t a(t-s)A_2u(s)\,ds$$
$$= f(t) - A_1y(t-h) - \int_{t-h}^0 a(t-s)A_2y(s)ds \equiv g(t),$$

$$(4.2)\quad u(0) = x$$

By the assumption (F.i) (II.iv) it is obvious that $g \in L^2(0,T;V^*)$. In view of (II.iv)

$$(4.3)\quad \int_0^h |A_1y(t-h)|^2 t\,dt = |A_1A^{-1}|^2 \int_{-h}^0 |Ay(s)|^2(s+h)\,ds < \infty.$$

With the aid of an integration by parts

$$(4.4)\quad \int_{t-h}^0 a(t-s)A_2y(s)ds$$
$$= A_2A^{-1}\{a(t)\int_{t-h}^0 Ay(\sigma)d\sigma - \int_{t-h}^0\int_{t-h}^s Ay(\sigma)d\sigma d_s a(t-s)\}.$$

By (II.iv), (II.v)

$$C_1 \equiv \sup_{-h<s<t\le 0} \left|\int_s^t Ay(\sigma)d\sigma\right| < \infty.$$

Hence, using (4.4)

$$(4.5)\quad \left|\int_{t-h}^0 a(t-s)A_2y(s)ds\right| \le C_1|A_2A^{-1}|\{|a(-t)| + V(a;t,h)\}.$$

From (F.i), (4.3), and (4.5) it follows that $g \in L^2(0,h;H,t\,dt)$. In view of (F.ii), (II.v), and (4.5)

$$\int_{+0}^{h} g(t)dt = \int_{+0}^{h} f(t)\,dt + A_1A^{-1}\int_{-h+0}^{0} Ay(s)\,ds + \int_{-h}^{0}\int_{t-h}^{0} a(t-s)A_2y(s)\,ds\,dt$$

exists in H. Hence, we can apply Theorem 1 to show the existence of a solution u of (4.1) and (4.2) such that

$$u \in L^2(0,h;V) \cap L^2(0,h;D(A),t\,dt),$$
$$\int_{+0}^{h} Au(t)\,dt \text{ exists in } H.$$

Suppose that it has been shown that a solution u exists in $[0,nh]$ for some integer n with $nh < T$ such that

$$u \in L^2(ih,(i+1)h;V) \cap L^2(ih,(i+1)h;D(A),(t-ih)\,dt), \tag{4.6}$$
$$\int_{ih+0}^{(i+1)h} Au(t)\,dt \in H \tag{4.7}$$

for $i = 0,1,2,\ldots,n-1$. For $nh < t < (n+1)h \wedge T$ the equation (1.3) is reduced to

$$\frac{d}{dt}u(t) + Au(t) + \int_{nh}^{t} a(t-s)A_2u(s)\,ds = f(t) - A_1u(t-h) - \int_{t-h}^{nh} a(t-s)A_2u(s)ds \equiv g(t). \tag{4.8}$$

From (4.6) with $i = n-1$ it readily follows that $g \in L^2(nh,(n+1)h \wedge T;V^*)$. Following the technique of the proof of (4.4) we get

$$\int_{t-h}^{nh} a(t-s)A_2u(s)\,ds = A_2A^{-1}\{a(t-nh)\int_{t-h}^{nh} Au(\sigma)\,d\sigma - \int_{t-h}^{nh}\int_{t-h}^{s} Au(\sigma)\,d\sigma\,d_sa(t-s)\}. \tag{4.9}$$

In view of (4.6), (4.7)

$$C_2 \equiv \sup_{(n-1)h \leq \tau < s \leq nh} \left| \int_\tau^s Au(\sigma)\, d\sigma \right| < \infty. \tag{4.10}$$

From (4.9) and (4.10) we get

$$\begin{aligned} &\left| \int_{t-h}^{nh} a(t-s) A_2 u(s)\, ds \right| \\ \leq & C_2 |A_2 A^{-1}| \{ |a(t-nh)| + V(a; t-h, nh) \}. \end{aligned} \tag{4.11}$$

By virtue of (4.6), (4.7) with $i = n-1$

$$\begin{aligned} &\int_{nh}^{(n+1)h \wedge T} |A_1 u(t-h)|^2 (t-nh)\, dt \\ = &\int_{(n-1)h}^{nh \wedge T} |A_1 u(t)|^2 (t-(n-1)h)\, dt < \infty, \end{aligned} \tag{4.12}$$

$$\int_{nh+0}^{(n+1)h \wedge T} A_1 u(t-h)\, dt = A_1 A^{-1} \int_{(n-1)h+0}^{nh \wedge T} Au(t)\, dt \in H. \tag{4.13}$$

With the aid of (F.ii), (4.11), (4.12), and (4.13) we get

$$\begin{aligned} &g \in L^2(nh, (n+1)h \wedge T; H, (t-nh)dt), \\ &\int_{nh+0}^{(n+1)h \wedge T} g(t)\, dt \in H. \end{aligned}$$

Hence, applying Theorem 1 to (4.8) in $[nh, (n+1)h \wedge T]$ we see that there exists a solution u of (4.8) satisfying the initial condition $u(nh+0) = u(nh-0)$ such that

$$\begin{aligned} u \in C([nh, (n+1)h \wedge T]; H) &\cap L^2(nh, (n+1)h \wedge T; V) \\ &\cap L^2(nh, (n+1)h \wedge T; D(A), (t-nh)\, dt) \end{aligned}$$

and

$$\int_{nh+0}^{(n+1)h \wedge T} Au(t)\, dt \quad \text{exists in } H.$$

Appendix

We give an example of H, V, f such that
(A.1) $f \in L^2(0,T;V^*) \cap L^2(0,T;H,tdt)$,
(A.2) $\int_{+0}^T f(t)dt$ exists in H
(A.3) $\int_0^T |f(t)|dt = \infty$.
Let Λ be the operator associated with the inner product $((\cdot,\cdot))$ of V:

$$(\Lambda u, v) = ((u,v)), \qquad u, v \in V.$$

Then, the realization of Λ in H is positive definite and self-adjoint. For $u_0 \in H$ set $u(t) = e^{-t\Lambda}u_0$. Then, it is easy to see that

$$f(t) = u'(t) = -\Lambda e^{-t\Lambda}u_0 \tag{a.1}$$

satisfies (A.1), and (A.2) (cf. Lemma 3.1).

It remains to choose H, V, u_0 so that the function $f(t)$ defined by (a.1) satisfies (A.3).

Let Ω be a bounded domain in $\mathbf{R}^n$ with smooth boundary. Let $H = L^2(\Omega)$, $V = H_0^1(\Omega)$. Then

$$((u,v)) = \sum_{n=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i} \cdot \overline{\frac{\partial v}{\partial x_i}}\, dx \tag{a.2}$$

is an inner product in $H_0^1(\Omega)$. The realization in $L^2(\Omega)$ of the operator associated with $(a.2)$ is

$$D(\Lambda) = H^2(\Omega) \cap H_0^1(\Omega), \quad \Lambda u = -\Delta u \quad \text{for } u \in D(\Lambda).$$

Denote the eigenvalues of Λ by λ_j, $j = 1, 2, \ldots$, and the corresponding orthonormal set of eigenfunctions by $\{\varphi_j\}$. We suppose that $\{\lambda_j\}$ are arranged in the increasing order and repeated according to the multiplicity.

For $\lambda \geq 0$ let $N(\lambda)$ be the number of the eigenvalues of Λ which do not exceed λ. It is known that

$$N(\lambda) = c_0\lambda^{n/2} + o(\lambda^{n/2})$$

as $\lambda \to \infty$, where c_0 is some positive constant. Hence, there exists a positive constant c such that

$$\lambda_j \geq c j^{2/n}, \quad j = 1,\ 2,\ \ldots, \tag{a.3}$$

$$(c\lambda)^{n/2} \leq N(\lambda) \leq (c^{-1}\lambda)^{n/2} \text{ for } \lambda \geq \lambda_1. \tag{a.4}$$

We use the following elementary fact:

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p} \begin{cases} < \infty, & \text{if } p > 1 \\ = \infty, & \text{if } p \leq 1. \end{cases} \tag{a.5}$$

Let

$$u_0 = \sum_{j=1}^{\infty} a_j \varphi_j, \quad a_j = (j+1)^{-1/2}\{\log(j+1)\}^{-2/3}.$$

In view of $(a.5)$ $\sum_{j=1}^{\infty} a_j^2 < \infty$. Hence, u_0 is an element of $L^2(\Omega)$. Put $u(t) = e^{-t\Lambda}u_0$. Then,

$$u'(t) = -\sum_{j=1}^{\infty} a_j \lambda_j e^{-\lambda_j t} \varphi_j.$$

In what follows we suppose $0 < t \leq \lambda_1^{-1}$. Since $\{a_j\}$ is a decreasing sequence

$$|u'(t)|^2 = \sum_{j=1}^{\infty} a_j^2 \lambda_j^2 e^{-2\lambda_j t} \geq \sum_{j=1}^{N(1/t)} \lambda_j^2 e^{-2\lambda_j t} a_{N(1/t)}^2. \tag{a.6}$$

Noting that $x^2 e^{-2xt}$ is an increasing function of x in the interval $[0, 1/t]$ and $cj^{2/n} \leq \lambda_j \leq 1/t$ for $j \leq N(1/t)$ in view of $(a.3)$ we get

$$\sum_{j=1}^{N(1/t)} \lambda_j^2 e^{-2\lambda_j t} \geq \sum_{j=1}^{N(1/t)} (cj^{2/n})^2 e^{-2cj^{2/n} t}. \tag{a.7}$$

Since $x^{4/n}e^{-2cx^{2/n}t}$ is an increasing function of x in $[0,(cj)^{-n/2}]$ and $N(1/t) \leq (ct)^{-n/2}$ in view of $(a.4)$

$$(a.8) \qquad \int_0^{N(1/t)} x^{4/n} e^{-2cx^{2/n}t} dx = \sum_{j=1}^{N(1/t)} \int_{j-1}^{j} x^{4/n} e^{-2cx^{2/n}t} dx \leq \sum_{j=1}^{N(1/t)} j^{4/n} e^{-2cj^{2/n}t}.$$

On the other hand noting $N(1/t) \geq (c/t)^{n/2}$ in view of $(a.4)$ and by the change of the variable $y = x^{2/n}t$

$$(a.9) \qquad \int_0^{N(1/t)} x^{4/n} e^{-2cx^{n/2}t} dx \geq \int_0^{(c/t)^{n/2}} x^{4/n} e^{-2cx^{2/n}t} dx = \frac{n}{2} t^{-2-n/2} \int_0^c y^{n/2+1} e^{-2cy} dy$$

Combining $(a.6)$, $(a.7)$, $(a.8)$, $(a.9)$ we get

$$(a.10) \qquad |u'(t)|^2 \geq \frac{n}{2} c^2 t^{-2-n/2} \int_0^c y^{n/2+1} e^{-2cy} dy \cdot a_{N(1/t)}^2.$$

Set $b(t) = [(ct)^{-n/2}]$, where $[\ \cdot\ ]$ means the integral part. By view of $(a.4)$ $N(1/t) \leq b(t)$. Since $\{a_j\}$ is a decreasing sequence, it follows from $(a.10)$ that

$$(a.11) \qquad |u'(t)| \geq c_1 t^{-1-n/4} a_{b(t)}$$

for some positive constant c_1. Hence, with the aid of the change of the variable $s = (ct)^{-n/2}$

$$\int_0^{\lambda_1} |u'(t)|\, dt \geq c_1 \int_0^{\lambda_1} t^{-1-n/4} a_{b(t)}\, dt$$
$$\geq \frac{2}{n} c_1 c^{n/4} \int_N^{\infty} s^{-1/2} a_{[s]}\, ds,$$

where $N = [(\lambda_1/c)^{n/2}] + 1$. As is easily seen

$$\int_N^{\infty} s^{-1/2} a_{[s]}\, ds = \sum_{j=N}^{\infty} \int_j^{j+1} s^{-1/2} a_{[s]}\, ds$$
$$\geq \sum_{j=N}^{\infty} (j+1)^{-1/2} a_j$$
$$= \sum_{j=N}^{\infty} (j+1)^{-1} \{\log(j+1)\}^{-2/3} = \infty$$

by virtue of $(a.5)$. Thus we conclude

$$\int_0^{\lambda_1} |f(t)|\, dt = \int_0^{\lambda_1} |u'(t)|\, dt = \infty.$$

References

1. V. Barbu, *Nonlinear semigroups and differential equations in Banach spaces*, Noordhoff International Publishing, 1976.
2. G. Di. Blasio, K. Kunisch and Sinestrari, *L^2-regularity for parabolic partial integrodifferential equations with delay in the highest-order derivatives*, J. Math.Anal. Appl. **102** (1984), 38–57.
3. ———, *Stability for abstract linear functional differential equations*, Israel J. Math. **50** (1985), 231–263.
4. H. Brezis, *Opérateus maximaux monotones et semigroupes de contractions dans les espaces de Hilbert*, North-Holland Publishing Company, Amsterdam, London, 1973.
5. M. G. Crandall and J. A. Nohel, *An abstract functional differential equation and a related nonlinear Volterra equation*, Israel J. Math. **29** (1978), 313–328.

6. S. O. Londen and J. A. Nohel, *An abstract nonlinear Volterra integro-differential equation*, J. Math. Anal. Appl. **64** (1978), 701–735.
7. J. L. Lions, *Équations differéntielles opérationnelles et problèmes aux limites*, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1961.
8. S. Nakagiri, *Pointwise completeness and degeneracy of functional differential equations in Banach spaces, I. General time delays*, J. Math. Anal. Appl. **127** (1987), 492–529.
9. ———, *Structural properties of functional differential equations in Banach spaces*, Osaka J. Math. **25** (1989), 353–398.
10. J. Prüss, *On resolvent operators for linear integro-differential equations of Volterra type*, J. Integral Equations **5** (1983), 211–236.
11. E. Sinestrari, *On a class of retarded partial differential equations*, Math. Z. **186** (1984), 223–246.
12. H. Tanabe, *On fundamental solution of differential equation with time delay in Banach space*, Proc. Japan Acad. **64** (1988), 131–134.
13. ———, *Equations of evolution*, Pitman-London, 1979.

Department of Mathematics
Dong-A University
Pusan 604–714, Korea
and
Department of Mathematic Education
ChinJu Teacher's College
Chinju 660-756, Korea