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CRITICAL POINTS OF REAL ENTIRE FUNCTIONS

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1. Introduction

This note is concerned with the zeros of a real entire function $f(z)$ and those of $f'(z)$. A real entire function is an entire function which assumes only real values on the real axis. Thus the zeros of a real entire function $f(z)$ are symmetrically located with respect to the real axis. In order to state concisely the background to our results as well as the results themselves, we introduce some terminologies.

Let $f(z)$ be a nonconstant real entire function. Suppose that ξ is a real zero of $f^{(l)}(z)$ of multiplicity m but not a zero of $f^{(l-1)}(z)$. That is

$$f^{(l-1)}(\xi) \neq 0,\ f^{(l)}(\xi) = f^{(l+1)}(\xi) = \cdots = f^{(l+m-1)}(\xi) = 0,\ f^{(l+m)}(\xi) \neq 0.$$

Put

$$k = \begin{cases} \dfrac{m}{2}, & \text{if } m \text{ is even,} \\ \dfrac{m+1}{2}, & \text{if } m \text{ is odd and } \quad f^{(l-1)}(\xi) f^{(l+m)}(\xi) > 0, \\ \dfrac{m-1}{2}, & \text{if } m \text{ is odd and } \quad f^{(l-1)}(\xi) f^{(l+m)}(\xi) < 0. \end{cases}$$

If $k > 0$ we shall say that ξ is a *critical zero* of $f^{(l)}(z)$ of the multiplicity k. Let $K(f^{(l)})$, $l = 1, 2, \cdots$, be the sum of the multiplicities of the critical zeros of $f^{(l)}(z)$, and let $K_T(f) = \sum_{l=1}^{\infty} K(f^{(l)})$. (If $f(z)$ is a constant function, then we set $K(f^{(l)}) = 0$, $l = 1, 2, \cdots$.) $K(f^{(l)})$ is called the number of critical zeros of $f^{(l)}(z)$, and $K_T(f)$ is called the total number of critical points of $f(z)$. On the other hand, $Z_C(f)$ will denote the number of nonreal zeros of $f(z)$, counting multiplicities.

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If $f(z)$ is a real polynomial, it is easy to see that

$$Z_C(f) - Z_C(f') = 2K(f'), \tag{1}$$

$$\lim_{l \to \infty} Z_C(f^{(l)}) = 0, \tag{2}$$

and hence we have

$$Z_C(f) = 2K_T(f). \tag{3}$$

The purpose of this note is to generalize (3) to a class of transcendental functions.

A real entire function $f(z)$ is said to be of *genus* 1^* if it can be expressed in the form

$$f(z) = e^{-\alpha z^2} g(z),$$

where $\alpha \geq 0$ and $g(z)$ is a real entire function of genus at most one. Thus, if $f(z)$ is of genus 1^*, then its genus may be 0 or 1 or 2, but in the last case it is only slightly higher than genus 1. If $f(z)$ is a real entire function of genus 1^* and if $Z_C(f) = 0$, then it is called a Laguerre–Pólya function, because a classical theorem of Laguerre and Pólya asserts that $f(z)$ is a Laguerre–Pólya function if and only if it can be uniformly approximated on compact sets in the plane by a sequence of real polynomials with only real zeros. (For a proof of this theorem see Levin [L, Chapter 8].) The class of all Laguerre–Pólya functions will be denoted by $\mathcal{LP}$, and the class of all real entire functions of genus 1^* which have finitely many nonreal zeros will be denoted by $\mathcal{LP}^*$. From the above fact and Rolle's theorem, it follows that if $f \in \mathcal{LP}^*$, then

$$Z_C(f') \leq Z_C(f). \tag{4}$$

Moreover, the classes $\mathcal{LP}$ and $\mathcal{LP}^*$ are closed under differentiation.

In 1930, Pólya [P1] showed that if $f \in \mathcal{LP}^*$, then (1) is true. In the same paper, he also conjectured the following proposition which is called the Pólya–Wiman conjecture.

THE PÓLYA–WIMAN CONJECTURE. *If $f \in \mathcal{LP}^*$, then (2) is true.*

This conjecture has been proved by Craven, Csordas, Smith and Kim [CCS1], [CCS2], [K]. Hence (3) is true for $f \in \mathcal{LP}^*$.

On the other hand, it is known that if a real entire function $f(z)$ is not of genus 1^*, then (1) and (2) are not true in general [HW1], [HW2], [P2], [S]. Therefore, if we want to proceed further, we must look at the functions of genus 1^* which have infinitely many nonreal zeros. But, as the function $f(z) = e^z + 1$ shows, (3) is not true for some functions of genus 1^*. Note that $f(z) = e^z + 1$ is of order 1 and its zero set is $\{(2n+1)\pi i \mid n = 0, \pm 1, \pm 2, \cdots\}$. Hence it is natural to consider the functions of order less than 1 or the functions whose (nonreal) zeros are sufficiently close to the real axis. In fact Pólya [P1] conjectured the following:

PÓLYA'S CONJECTURE (1930). If $f(z)$ is a real entire function of genus 0, then (3) is true.

It seems that this conjecture is open since 1930.

Instead of restricting the genus (or order), we will restrict the zero set and obtain the following: Let $\rho_C(f)$ (resp. $\rho_K(f')$) be the convergence exponent of the nonreal zeros of $f(z)$ (resp. the critical zeros of $f'(z)$), that is

$$\rho_C(f) = \overline{\lim_{r\to\infty}} \frac{\log n_C(r)}{\log r}, \quad \rho_K(f') = \overline{\lim_{r\to\infty}} \frac{\log n_K(r)}{\log r},$$

where $n_C(r)$ (resp. $n_K(r)$) is the number of nonreal zeros of $f(z)$ (resp. the critical zeros of $f'(z)$) in $|z| \le r$. Then we have;

THEOREM 1. *If $f(z)$ is of genus 1^* and if $f(z)$ has no zeros outside an infinite strip $|\mathrm{Im}\, z| \le A$, $A > 0$, then*

$$\rho_C(f) = \max\{\rho_C(f'), \rho_K(f')\}. \tag{5}$$

THEOREM 2. *If $f(z)$ is of genus 1^* and if $f(z)$ has no zeros outside an infinite strip $|\mathrm{Im}\, z| \le A$, $A > 0$, then (2) implies (3).*

REMARK. regarded as a generalization of (4).

(b) If $\rho_C(f) > 0$, then Theorem 2 is an immediate consequence of Theorem 1. However Theorem 2 includes the case that $\rho_C(f) = 0$.

2. Preliminaries

Let $f(z)$ be a nonconstant real entire function. Enumerate the real zeros of $f(z)$ as follows:

$$\cdots \le a_{k-1} \le a_k \le a_{k+1} \le \cdots$$

$$(-\infty \le \alpha \le k \le \omega \le +\infty, \quad k \text{ finite }).$$

(In this sequence, a real zero of multiplicity m must appear exactly m times.) According to Rolle's theorem, we can find a sequence $\{b_k\}$ of real zeros of $f'(z)$ which satisfies

$$a_k \le b_k \le a_{k+1} \qquad \text{for all } k < \omega.$$

Note that $f'(z)$ can have real zeros which do not appear in the sequence $\{b_k\}$. These zeros will be called the *extra zeros* of $f'(z)$, and the number of extra zeros of $f'(z)$ will be denoted by $E(f')$. It is easy to see that

$$(6) \qquad 2K(f) \le E(f) \le 2K(f) + 2.$$

Now set

$$(7) \qquad \psi(z) = \prod_{|a_k|,|b_k| \le 1} \left(\frac{z - b_k}{z - a_k} \right) \prod_{|a_k|,|b_k| > 1} \left(\frac{1 - z/b_k}{1 - z/a_k} \right).$$

(If $f(z)$ has no real zeros at all, set $\psi(z) \equiv 1$, and if it has only one real zero a_0, set $\psi(z) = (z - a_0)^{-1}$.) It is well known that the product (7) converges uniformly on any compact set not containing the points a_k [L, p.308]. Moreover, we have the following:

LEMMA 1. *The function* $w = \psi(z)$ *maps the upper half plane* $\operatorname{Im} z > 0$ *into a half plane.*

Proof. See [L p.308]. □

This meromorphic function $\psi(z)$ will be called the *Levin function* of $f(z)$.

For each nonnegative real number A let $\mathcal{LP}^A$ be the class of real entire functions of genus 1^* which have no zeros outside the infinite strip $|\operatorname{Im} z| \leq A$. It is known that $f \in \mathcal{LP}^A$ if and only if $f(z)$ can be uniformly approximated on compact sets in the plane by a sequence of real polynomials all of whose zeros lie in the infinite strip $|\operatorname{Im} z| \leq A$ [L, Chapter 8]. In particular, the class $\mathcal{LP}^A$ is closed under differentiation.

Note that if $f \in \mathcal{LP}^A$, then $f(z)$ can be expressed in the form

$$f(z) = cz^n e^{-\alpha z^2+\beta z} \prod_k (1 - \frac{z}{a_k}) e^{\frac{z}{a_k}} \prod_j (1 - \frac{z}{c_j})(1 - \frac{z}{\bar{c}_j}) e^{(\frac{1}{c_j}+\frac{1}{\bar{c}_j})z},$$

where n is a nonnegative integer, $\alpha \geq 0$, c, β, and a_k are real, $|\operatorname{Im} c_j| \leq A$, $\sum |a_k|^{-2} < \infty$ and $\sum |c_j|^{-2} < \infty$. Therefore the logarithmic derivative of $f(z)$ is given by

$$\frac{f'(z)}{f(z)} = \frac{n}{z} - 2\alpha z + \beta + \sum_k \left(\frac{1}{z - a_k} + \frac{1}{a_k}\right) + \sum_j \left(\frac{1}{z - c_j} + \frac{1}{z - \bar{c}_j} + \frac{2\operatorname{Re} c_j}{|c_j|^2}\right),$$

and hence we have

LEMMA 2. *If* $f \in \mathcal{LP}^A$, *then the function* $w = f'(z)/f(z)$ *maps the half plane* $\operatorname{Im} z > A$ *into the lower half plane* $\operatorname{Im} w < 0$.

In the proof our theorems we will use the following lemmas.

LEMMA 3 (CARATHEODORY INEQUALITY). *Let* $w = f(z)$ *be an analytic function defined on the upper half plane* $\operatorname{Im} z > 0$. *If* $w = f(z)$ *maps the upper half plane* $\operatorname{Im} z > 0$ *into the upper half plane* $\operatorname{Im} w > 0$, *then*

$$\frac{1}{5}|f(i)|\frac{\sin\theta}{r} < |f(re^{i\theta})| < 5|f(i)|\frac{r}{\sin\theta}, \qquad (r > 1, \quad 0 < \theta < \pi).$$

Proof. See [L, p.18]. □

LEMMA 4. *Let $f \in \mathcal{LP}^A$ and let ρ be the convergence exponent of the zeros of $f(z)$. If the genus of $f(z)$ is less than 2, then*

$$\overline{\lim_{r\to\infty}} \frac{\log\log|f(ir)|}{\log r} = \rho.$$

Proof. The order of the even function $g(z) = f(z)f(-z)$ is exactly ρ. Let $h(z) = g(\sqrt{z})$. Then $h(z)$ is a real entire function of order $\rho/2$ and the zeros of $h(z)$ lie in the region $\{x + iy \mid x \geq y^2/(2A)^2 - A^2\}$. Therefore

$$\begin{aligned}\rho &= \overline{\lim_{r\to\infty}} \frac{\log\log\left(\max_{|z|=r^2}|h(z)|\right)}{\log r}\\ &= \overline{\lim_{r\to\infty}} \frac{\log\log h(-r^2)}{\log r}\\ &= \overline{\lim_{r\to\infty}} \frac{\log\log g(ir)}{\log r}\\ &= \overline{\lim_{r\to\infty}} \frac{\log\log|f(ir)|}{\log r} \qquad \square\end{aligned}$$

3. Proof of the Theorems

Proof of Theorem 1. Let $f \in \mathcal{LP}^A$, and let $\Pi(z)$ and $\Pi_1(z)$ be the canonical products of the nonreal zeros of $f(z)$ and $f'(z)$, respectively. The logarithmic derivative of $f(z)$ can be expressed in the form

$$\frac{f'(z)}{f(z)} = \frac{\Pi_1(z)}{\Pi(z)}\psi(z)\phi(z), \tag{8}$$

where $\psi(z)$ is the Levin function of $f(z)$ and $\phi(z)$ is a real entire function. It is clear that $\Pi(z)$, $\Pi_1(z)$ and $\phi(z)$ are of genus at most 1, and the zeros of $\phi(z)$ are exactly the extra zeros of $f'(z)$.

From Lemmas 1, 2 and 3, there are positive constants C_1 and C_2 such that

$$C_1 r^{-2} < \left|\frac{f'(ir)}{f(ir)}\frac{1}{\psi(ir)}\right| < C_2 r^2$$

for all sufficiently large r, and hence (8) gives

$$C_1 r^{-2}|\Pi(ir)| < |\Pi_1(ir)\phi(ir)| < C_2 r^2 |\Pi(ir)|.$$

Now Lemma 4 gives

$$\begin{aligned} \rho_C(f) &= \overline{\lim_{r\to\infty}} \frac{\log\log|\Pi(ir)|}{\log r} \\ &= \overline{\lim_{r\to\infty}} \frac{\log\log|\Pi_1(ir)\phi(ir)|}{\log r} \\ &= \max\{\rho_C(f'), \rho_E(f')\}, \end{aligned} \tag{9}$$

where $\rho_E(f')$ is the convergence exponent of the extra zeros of $f'(z)$. It is clear that $\rho_E(f') = \rho_K(f')$, and (9) gives the desired result. □

Proof of Theorem 2. Let $f \in \mathcal{LP}^A$ and assume that $\lim_{n\to\infty} Z_C(f^{(n)}) = 0$. Since (1) is true for all $f \in \mathcal{LP}^*$, we may assume, without loss of generality, that $Z_C(f) = \infty$. Then there must be an integer l such that $Z_C(f^{(l)}) = \infty$ and $Z_C(f^{(l+1)}) < \infty$. It suffices to show that $f^{(l+1)}(z)$ has infinitely many critical zeros.

To get a contradiction, assume that $f^{(l+1)}(z)$ has only a finite number of critical zeros. From (6) $f^{(l+1)}(z)$ has only a finite number of extra zeros and hence we have

$$\frac{f^{(l+1)}(z)}{f^{(l)}(z)} = \frac{e^{\gamma z}P(z)\psi(z)}{\Pi(z)}, \tag{10}$$

where γ is a real constant, $P(z)$ is a real polynomial, $\psi(z)$ is the Levin function of $f^{(l)}(z)$ and $\Pi(z)$ is the canonical product of the nonreal zeros of $f^{(l)}(z)$.

From Lemmas 1, 2 and 3, there is a positive constant C such that

$$Cr^{-2} < \left| \frac{f^{(l+1)}(z)}{f^{(l)}(z)} \frac{1}{\psi(z)} \right|$$

for all sufficiently large r. Now (10) gives

$$|\Pi(ir)| < C^{-1} r^2 |P(ir)|,$$

which is impossible since $\Pi(z)$ is an infinite product all of whose zeros lie in the infinite strip $|\operatorname{Im} z| \le A$. This contradiction shows that $f^{(l)}(z)$ must have infinitely many critical zeros. □

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