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ESTIMATIONS OF THE FOURIER–LAPLACE TRANSFORMS OF ${}_{\phi}\mathrm{K}_1'$

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The Fourier-Laplace image $F_u(\xi+i\eta)$ of $u(x)$ is defined by the Fourier transforms of $e^{\eta x}u(x)$. Gel'fand-Shilov [3] shows that the Fourier-Laplace image of a class of C^∞-functions, any element of which can be estimated by $C\exp(-\phi(1\pm\varepsilon)x)$ together with its all derivatives, coincides with the class of entire function $F(\xi+i\eta)$, which satisfies the estimate $|(\xi+i\eta)^k F(\xi+i\eta)| \le C_k \exp\psi(\eta)$ for all k, where $\psi(\eta)$ is the dual function of the N-function $\phi(x)$ in the sense of Young [2]. Hayakawa [4] shows that the Fourier-Laplace image of ${}_{\phi}S'$ which consists of all distributions of the type $\exp(-\phi(x))u(x)$, $u(x)\in S'$, is nearly equal to the class of entire functions $F(\xi+i\eta)$ which satisfy the estimate $|F(\xi+i\eta)| \le C(1+|\xi+i\eta|)^k \exp\psi(\eta)$ for an integer k. In this paper we consider this problem in ${}_{\phi}K_1'$.

By $K_{1,r}$, where $r>0$, we denote the space of all complex valued C^∞-functions f on R such that, for any $j=0,1,2,\ldots,p\in N$

$$e^{r_j|x|}|D^p f(x)| < \infty$$

where $0<r_j<r_{j+1}<\cdots$, $r_j\to r$. The topology in $K_{1,r}$ is defined by the semi-norms

$$q_{j,p}(f) = \sup(e^{r_j|x|}|D^p f(x)|), \quad j=0,1,2,\ldots,p\in N.$$

From the definition it follows that $K_{1,r}\subset K_{1,s}$ whenever $r>s$. When $r=\infty$ one gets the space $K_{1,\infty}$ which will be also denoted by K_1 and is the projective limit of the $K_{1,r}$'s.

Let $K_{1,r}'$ be the dual of $K_{1,r}$. The dual K_1' of K_1 is the space of distributions of exponential growth. Since the space $\mathcal{D}$ is dense in $K_{1,r}$ for

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all $r > 0$, it follows that $K'_{1,r}$ is a subspace of the space $\mathcal{D}'$ of Schwartz's distributions. If $r > s > 0$, then

$$K'_{1,s} \subset K'_{1,r} \subset K'_1.$$

K'_1 is the inductive limit of the $K'_{1,r}$'s.

$E_{1,r}$, with $r > 0$, will denote the space of complex valued functions $g(z)$ which are analytic in the strip $V_r = \{z \in C; |\mathrm{Im}\, z_j| < r,\ j = 0,1,2,\dots,n\}$ and rapidly decreasing in any narrower strip V_{r_j}, $0 < r_j < r_{j-1}$, $r_j \to r$, i.e., for any $j = 0,1,2,\dots$ and for any $k = 0,1,2,\dots$

$$\nu_k(g) = \sup_{z\in V_{r_j}} (1+|z|)^k |g(z)| < \infty.$$

When $r = \infty$ the space $E_{1,\infty}$, denoted in this work by E_1, is the space of all entire functions which are rapidly decreasing in any horizontal strip. In other words an entire function f is in E_1 if and only if for every $k = 0,1,2,\dots$

$$\nu_k(g) = \sup_{z\in V_k} (1+|z|)^k |g(z)| < \infty.$$

The topology of $E_{1,r}$, $r > 0$ will be the one generated by the seminorms ν_k, $k = 0,1,2,\cdots$. The Fourier transform is a topological isomorphism from $K_{1,r}$ onto $E_{1,r}$ [1]. The dual $E'_{1,r}$ of $E_{1,r}$ is the space of Fourier transforms of distributions in $K'_{1,r}$. The Fourier transform $\hat{u}$ of $u \in K'_{1,r}$ is defined by the Parseval-Plancherel formula

$$\langle \hat{u}, \hat{f} \rangle = \frac{1}{2\pi} \langle u, \check{f} \rangle$$

where $\check{f}(x) = f(-x)$, $f \in K_{1,r}$.

For $T \in E'_{1,r}$, the derivative $\frac{d}{dz} T$ is defined by the formula

$$\langle \frac{d}{dz} T,\ g \rangle = -\langle T,\ \frac{dg}{dz} \rangle, \quad g \in E_{1,r}.$$

From the continuity of the operator $\dfrac{d}{dz}$ on $E_{1,r}$, it follows that $\dfrac{d}{dz}T \in E_{1,r}'$ whenever $T \in E_{1,r}'$. We also have $S' \subset K_1'$ where S' is the space of tempered distributions. By taking the Fourier transform and noticing that the Fourier transform is an automorphism on S', we obtain that $S' \subset E_1'$.

The following lemma characterizes the elements of K_1'.

LEMMA 1. [5]. *A distribution $T \in \mathcal{D}'$ is in K_1' if and only if T can be represented in the form $T = D^p[e^{k|x|}F(x)]$ where $p \in N$, $k \in R$ and F is a bounded continuous function on R.*

For an N-function ϕ which belongs to O_M, we define spaces ${}_{\phi}K_1'$ and ${}_{\phi,\Omega}K_1'$ as follows;

$$
\begin{aligned}
{}_{\phi}K_1' &= \{u \in \mathcal{D}' \mid \exp(\phi(x))u(x) \in K_1'\} \\
{}_{\phi,\Omega}K_1' &= \{u \in \mathcal{D}' \mid \exp(\lambda x + \phi(x))u(x) \in K_1' \text{ for all } \lambda \text{ in } \Omega\}.
\end{aligned}
$$

THEOREM 2. *For any positive real ε, there exist k, N and C_ε such that the Fourier-Laplace transform $F_u(\xi + i\eta)$ of $u \in {}_{\phi}K_1'$ satisfies*

$$|F_u(\xi + i\eta)| \leq C_\varepsilon(1 + |\xi + i\eta|)^N \exp(\psi(\eta \pm k \pm \varepsilon)$$

Proof. Since $\exp(-\phi(x) + \eta x)$ belongs to S, $\exp(\eta x)u(x)$ belongs to K_1' and hence we can consider the Fourier-Laplace transform of $u \in {}_{\phi}K_1'$.

$$F_u(\xi + i\eta) = {}_{K_1'}\langle \exp(\phi(u))u(x), \exp(-\phi(x) - i(\xi + i\eta)x\rangle_{K_1}.$$

By Lemma 1, $\exp(\phi(x))u(x) = D^p[e^{k|x|}F(x)]$ where F is a bounded continuous function. Thus we have

$$
\begin{aligned}
F_u(\xi + i\eta) &= {}_{K_1'}\langle \exp(k|x|)F(x),\ (-D)^p \exp(-\phi(x) - i\zeta x)\rangle_{K_1} \\
&= \int_R \exp(k|x|)F(x)(-D)^p \exp(-\phi(x) - i(\xi + i\eta)x)dx.
\end{aligned}
$$

Using the Leibniz formula,

$$(-D)^p \exp(-\phi(x) - i\zeta x) = \sum_{\ell=0}^{p} P_{p,\ell}(\phi', \dots, \phi^{(\ell)})\zeta^\ell \exp(-\phi(x) - i\zeta x)$$

where $P_{p,\ell}(x_1, \dots, x_\ell)$ is a polynomial in $x = (x_1, \dots, x_\ell)$. Thus

$$\begin{aligned}
|F_u(\xi + i\eta)| &< \int_{-\infty}^{\infty} \exp(k|x|) \sup |F(x)| \sum_{\ell=0}^{p} C_\ell |\zeta|^\ell (1 + x^2)^{n_{p,\ell}} \\
&\quad \times \exp(-\phi(x) + \eta x) dx \\
&\leq \sum_{\ell=0}^{p} C \cdot C_\ell |\zeta|^\ell \int_0^\infty \exp(-\phi(x) + (\eta + k)x + \varepsilon x) \exp(-\varepsilon x) dx \\
&\quad + \sum_{\ell=0}^{p} C \cdot C_\ell |\zeta|^\ell \int_{-\infty}^{0} \exp(-\phi(x) + (\eta - k)x - \varepsilon x) \exp(\varepsilon x) dx \\
&\leq C_\varepsilon (1 + |\zeta|)^N \exp(\psi(\eta \pm k \pm \varepsilon)).
\end{aligned}$$

COROLLARY 3. *For any λ in Ω and $\varepsilon > 0$, there exists $N_{\lambda,\varepsilon}$ and $C_{\lambda,\varepsilon}$ such that the Fourier-Laplace transform $F_u(\xi + i\eta)$ of $u \in {}_{\phi,\Omega}K_1'$ satisfies*

$$|F_u(\xi + i\eta)| \leq C_{\lambda,\varepsilon}(1 + |\xi + i\eta|)^{N_{\lambda,\varepsilon}} \exp(\psi(\eta \pm \lambda \pm k \pm \varepsilon)).$$

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