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NUMERICAL RESULTS ON THE EIGENVALUE DISTRIBUTION OF THE MATRIX $S_h^{-1}C_h$

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1. Introduction

Let $I = [0, 1]$ be the unit interval in R. Consider the simple differential equation of the form

$$\begin{aligned} Lu &= -u'' \\ u(0) &= 0, \quad u'(1) = 0. \end{aligned} \tag{1.1}$$

We know that in the conforming finite element discretization of

$$Lu = f, \qquad f \text{ in } L^2(I), \tag{1.2}$$

we have the linear system $S_hU_h = F_h$ where S_h is called as the stiffness matrix associated with a particular choice of basis for the finite element space $\hat{S}$.

Let $\{f_i : i = 1, \cdots, kn\}$ be a given basis for $\hat{S}$. Following [2], we partition [0, 1] into uniform subintervals $I_j = [x_{j-1}, x_j]$ for $j = 1, \cdots, n$ such that $0 = x_0 < x_1 < \cdots < x_n = 1$.

Now consider

$$-u''(x) = 0, \qquad u(0) = u'(1) = 0. \tag{1.3}$$

The collocation version of (1.3) is that for any $u_h = a_1f_1 + \cdots + a_{kn}f_{kn}$ in $\hat{S}$

$$a_1f_1''(t_j) + \cdots + a_{kn}f_{kn}''(t_j) = 0. \tag{1.4}$$

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where k is the number of quadrature points on I_j and t_j is the quadrature point of the Chebyshev-Gauss type quadrature.

We define $A_{jr} = f_r(t_j)$, $B_{jr} = -f_r''(t_j)$, and $W_{jr} = \text{diag}(w_j)$ where w_j is the weight corresponding to t_j. The matrix form of (1.4) is $B_h a = 0$, where a is the coefficient vector of u_h. This can be written as $A_h^t W_h B_h a = 0$. We call $C_h = A_h^t W_h B_h$ as the collocation matrix.

In this paper, we investigate the eigenvalue distibution of $S_h^{-1} C_h$ $(0 < h < 1)$. The numerical results show that the eigenvalues of the $S_h^{-1} C_h$ are nearly bounded when the Chebyshev–Gauss type quadrature nodes and weights are used.

2. Quadratures

Let (,) be the L^2 inner product defined on $\hat{S}$. Then we can formulate (1.2) as follows:

Find u_h in $\hat{S}$ such that

$$(u_h', v_h') = (f, v_h) \qquad \text{for all } v_h \text{ in } \hat{S}. \tag{2.1}$$

Applying integration by parts to (1.3), we have $(u_h', v_h') = 0$ which makes us define the so-called stiffness matrix

$$S_h(i,j) = (f_i', f_j'). \tag{2.2}$$

Note that the above stiffness matrix (2.2) is a positive definite matrix. See [3].

In [1] there are explicit formulas for the quadrature points and weights: For the Chebyshev–Gauss quadrature, the quadrature points and weights are given respectively by

$$\begin{aligned} x_j &= \cos\frac{(2j+1)\pi}{2n+2}, \\ w_j &= \frac{\pi}{n+1}, \qquad j = 0, \cdots, n. \end{aligned} \tag{2.3}$$

For the Chebyshev–Gauss–Radau quadrature, the quadrature points and weights are given respectively by

$$x_j = \cos\frac{2\pi j}{2n+1},$$

$$w_j = \begin{cases} \dfrac{\pi}{2n+1}, \; j = 0, \\ \dfrac{2\pi}{2n+2}, \; j = 1, \cdots, n. \end{cases} \tag{2.4}$$

For the Chebyshev–Gauss–Lobotta quadrature, the quadrature points and weights are given respectively by

$$x_j = \cos\frac{2\pi j}{n},$$

$$w_j = \begin{cases} \dfrac{\pi}{2n}, \; j = 0, \; n, \\ \dfrac{\pi}{n}, \; j = 1, \cdots, n-1. \end{cases} \tag{2.5}$$

3. Numerical results

For the computation, we use the cubic spline basis functions s and v on $\hat{S}[-1, 1]$ such that

(3.1)
$$s(-1) = s(0) = s(1) = 0, \quad s'(-1) = s'(1) = 0, \quad s'(0) = 1.$$

(3.2)
$$v(0) = 1, \quad v(-1) = v(1) = 0, \quad v'(-1) = v'(0) = v'(1) = 0.$$

We transform s and v linearly on each interval I_j so that we have $2n$ basis functions on I. Therefore each matrix mentioned is a $2n$-by-$2n$ matrix. Also we construct the collocation matrix with respect to this basis and the quadrature points and corresponding weights on I_j by linear transformation. See [4]. The following are computational results when $h = .1$, $.05$, $.025$, $.0125$ using MATLAB (1990 version).

Case 1. Using Chebyshev–Gauss quadrature (2.3)

Result 3.1 : The eigenvalues of $S_h^{-1}C_h$ are bounded. See figure 1. Both upper and lower bounds are positive. Moreover, the lower bound is bigger than 2. The collocation matrix C_h can also be shown numerically to have positive eigenvalues.

Case 2. Using Chebyshev–Gauss–Radau quadrature (2.4)

Result 3.2 : The eigenvalues of $S_h^{-1}C_h$ are bounded. See figure 2.

Case 3. Using Chebyshev–Gauss–Lobotta quadrature (2.5)

Result 3.3 : The lower bounds of eigenvalues of $S_h^{-1}C_h$ is 0 so that C_h has 0 eigenvalues. The multiplicity of eigenvalue 0 is half of the dimension of C_h. The upper bound of eigenvalues of $S_h^{-1}C_h$ is less than 8. See figure 3.

Eigenvalue distribution of $S_h^{-1}C_h$

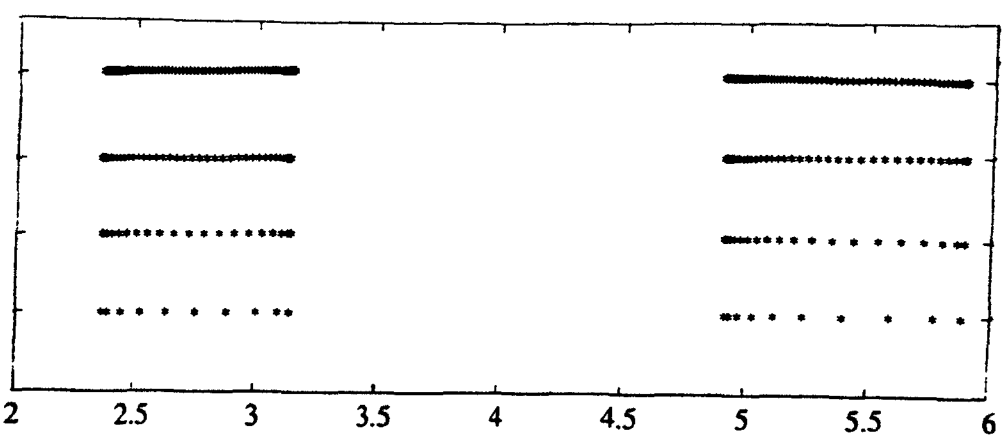

Fig 1. h= .1, .05, .025, .0125 in case 1.

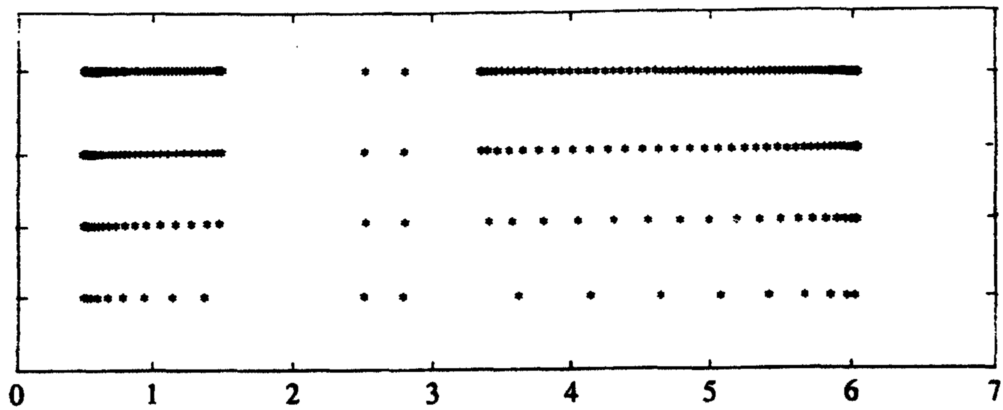

Fig 2. h= .1, .05, .025, .0125 in case 2.

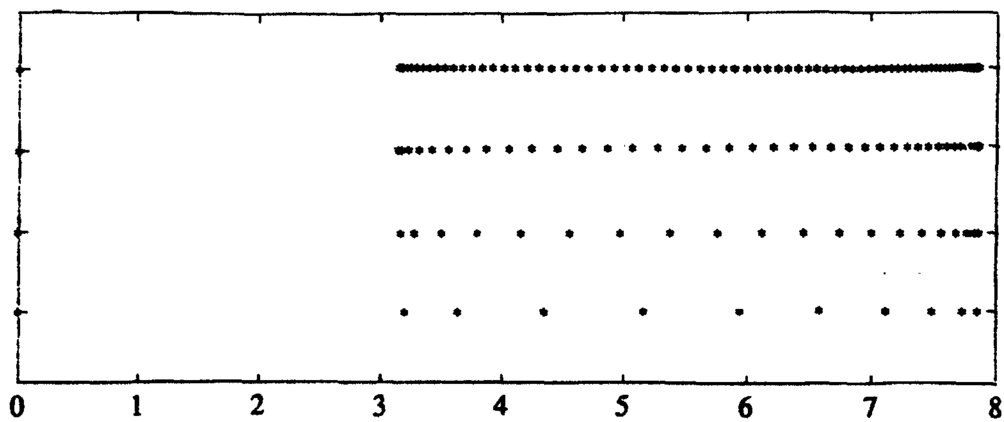

Fig 3. h= .1, .05, .025, .0125 in case 3.

4. Comments

The results can be shown numerically to hold for any mesh size h.

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