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FINITELY RELATIVE INJECTIVITY AND TORSION THEORIES

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Since G. Azumaya has introduced the notions of relative injective modules, the relative injectivity of certain classes of modules are extensively studied by many authors. Also, these are intimately related to the dual notion of relative projective modules. As we can see in [6], the injectivity of simple torsion modules has some important relations with special rings, such as V-rings and GV-rings. Hence we study the finite injectivity relative to a torsion theory. Throughout this paper, R means an associative ring with identity and all modules are unitary left modules. As usual R-**Mod** denotes the category of all left R-modules over a given ring R. For fundamental defnitions and results related to torsion theories, we refer to [4] and [5]. Let τ be a left exact preradical in R-**Mod**, $\mathcal{T}_\tau = \{ M \in R\text{-}\mathbf{Mod} \,|\, \tau(M) = M \}$ the associated pretorsion class, $\mathcal{L}_\tau = \{ I \,|\, I \text{ a left ideal of } R \text{ with } R/I \in \mathcal{T}_\tau \}$ the corresponding left linear topology on R and $(\mathcal{T}_\tau, \mathcal{F}_\tau)$ the torsion theory generated by $\mathcal{L}_\tau$. An R-module M is said to be *τ-injective* if $\mathrm{Ext}_R(R/I, M) = 0$ for all $I \in \mathcal{L}_\tau$. It is known that this is equivalent to the condition that given any exact sequence $0 \to A' \xrightarrow{i} A \to A'' \to 0$ of R-modules with $A'' \in \mathcal{T}_\tau$ and any map $\phi : A' \to M$, there exists a map $\psi : A \to M$ satisfying $\psi \circ i = \phi$ (see [1]).

In this note, we shall generalize τ-injective modules and obtain some results on finitely relative injectivity in the setup of 'torsion theories'. An equivalent condition for simple torsion modules to be finitely τ-injective relative to a torsion theory $(\mathcal{T}_\tau, \mathcal{F}_\tau)$ is given.

We begin with the following definition:

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DEFINITION 1. (1) An R-module M is called *finitely τ-injective* (in short, *f-τ-injective*) if $\mathrm{Ext}_R(R/I, M) = 0$ for every finitely generated $I \in \mathcal{L}_\tau$.

(2) An R-module M is called *strongly f-τ-injective* if given any exact sequence $0 \to A' \xrightarrow{i} A \to A'' \to 0$ of R-modules with $A'' \in \mathcal{T}_\tau$ and A' finitely generated, and any map $\phi : A' \to M$, there exists a map $\psi : A \to M$ satisfying $\psi \circ i = \phi$.

REMARKS. (1) Every τ-injective module is (strongly) f-τ-injective.

(2) Any direct sum of (strongly) f-τ-injective modules is again (strongly) f-τ-injective.

(3) If an R-module M is strongly f-τ-injective, then it is f-τ-injective. But the converse does not hold. In fact, if we set $\tau(M) = M$ for every R-module M, then τ is a left exact radical, $\mathcal{T}_\tau = R$-**Mod**, and $\mathcal{L}_\tau$ is the set of all left ideals of R. Hence an R-module M is f-R-injective if and only if $\mathrm{Ext}^1_R(R/I, M) = 0$ for all finitely generated $I \in \mathcal{L}_\tau$. But f-R-injectivity of M does not imply finite injectivity of M as we see in the following example: let $R = \prod_{p \in \mathcal{P}} Z/pZ$, where Z is the ring of integers and $\mathcal{P}$ is the set of positive prime integers, and let $T = \sum_{p \in \mathcal{P}} Z/pZ$. Then R/T is f-R-injective but it is not finitely injective. Hence f-τ-injectivity of M does not imply strong f-τ-injectivity of M by Lemma 2 below.

Now if τ is a left exact radical in R-**Mod**, then the associated torsion class $\mathcal{T}_\tau$ is hereditary (i.e., closed under submodules) and the corresponding topology $\mathcal{L}_\tau$ on R is a left Gabriel topology. A hereditary torsion theory is said to be *stable* if its torsion class is closed under injective envelopes. Relative to a stable torsion theory, strongly f-τ-injectivity is equivalent to finite injectivity as following:

LEMMA 2. *Assume that $\mathcal{T}_\tau$ is stable. Then $M \in \mathcal{T}_\tau$ is strongly f-τ-injective if and only if M is finitely injective.*

Proof. Let M_0 be a finitely generated submodule of $M \in \mathcal{T}_\tau$. Then $M_0 \in \mathcal{T}_\tau$ since $\mathcal{T}_\tau$ is hereditary. Hence $E(M_0) \in \mathcal{T}_\tau$, $E(M_0)/M_0 \in \mathcal{T}_\tau$, and the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_0 & \xrightarrow[i]{} & E(M_0) & \longrightarrow & E(M_0)/M_0 & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & & & & & \\ & & M & & & & & & \end{array}$$

where f and i are inclusion maps, yields a map $g : E(M_0) \to M$ such that $g \circ i = f$. Since f is monic and M_0 is essential in $E(M_0)$, g is a monomorphism. Thus M contains an injective envelope of each of finitely generated submodules, and hence M is finitely injective by Proposition 3.3 [3]. The converse is trivial and it completes the proof.

Note that $M \in \mathcal{T}_\tau$ is τ-injective if and only if M is injective when $\mathcal{T}_\tau$ is stable (see [6]). From this fact, the above Lemma and Corollary 3.4 [3] induce the following:

COROLLARY 3. *Assume that $\mathcal{T}_\tau$ is stable. Then a finitely generated module $M \in \mathcal{T}_\tau$ is strongly f-τ-injective if and only if M is τ-injective.*

Now we consider the conditions under which f-τ-injective modules are τ-injective. We first need the following lemma.

LEMMA 4. *For every directed family $(L_i)_I$ of (strongly) f-τ-injective submodules of an R-module M, the direct limit $\sum_I L_i$ is also (strongly) f-τ-injective.*

Proof. Consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A/A' & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & & & & & \\ & & \sum_I Li & & & & & & \end{array}$$

where A' is finitely generated and $A/A' \in \mathcal{T}_\tau$. Since A' is finitely generated, f carries it into some L_j. Since L_j is (strongly) f-τ-injective, the existence of $g : A \to L_j$ gives an extension $h : A \to \sum_I L_i$ of f.

THEOREM 5. *For a left exact radical τ in R-**Mod** and the corresponding left Gabriel topology $\mathcal{L}_\tau$ on R, the following statements are equivalent:*

(1) Every $I \in \mathcal{L}_\tau$ is finitely generated.

(2) A direct limit of τ-injective submodules of a given module is τ-injective.

(3) Every (strongly) f-τ-injective module is τ-injective.

Proof. (1) $\Rightarrow$ (3) is obvious and (2) $\Rightarrow$ (1) is contained in Theorem 2 [2]. By Lemma 4 and the hypothesis, $\sum_I L_i$ is τ-injective for every directed family $(L_i)_I$ of τ-injective submodules of a given module M. Thus (3) $\Rightarrow$ (2) holds. This completes the proof.

If we let $\tau(M) = M$ for every R-module M, then the corresponding torsion theory is stable. Hence Lemma 4 and Theorem 5 lead us to the following corollary.

COROLLARY 6. *(1) Every finitely injective R-module is injective if and only if R is left noetherian.*

(2) Every f-R-injective R-module is injective if and only if R is left noetherian.

Next, we consider finitely relative injectivity of simple torsion modules. For any proper left ideal I of R, let I^* denote the intersection of maximal left ideals of R containing I.

THEOREM 7. *For a left exact radical τ in R-**Mod**, the following statements are equivalent:*

(1) Every simple module in $\mathcal{T}_\tau$ is f-τ-injective.

(2) $I^* \neq K^*$ for every maximal ideal $K \in \mathcal{L}_\tau$ of a finitely generated proper ideal $I \in \mathcal{L}_\tau$.

Proof. (1) $\Rightarrow$ (2): Suppose that there exist a finitely generated $I \in \mathcal{L}_\tau$ and a maximal ideal $K \in \mathcal{L}_\tau$ of I such that $I^* = K^*$. Then $I/K \in \mathcal{T}_\tau$ since $R/K \in \mathcal{T}_\tau$ and $\mathcal{T}_\tau$ is hereditary. Since I/K is simple, $\mathrm{Ext}^1_R(R/I, I/K) = 0$. Hence there exists an extension $g : R \to I/K$ of the natural projection $f : I \to I/K$. Let h be the restriction of g to I^*. Then Ker h contains K and Ker $h \subseteq N^* = K^*$. So (Ker $h)^* = K^*$ since $(K^*)^* = K^*$. Now Ker g is a maximal left ideal of R, (Ker g) $\cap I^* =$ Ker

h and $I^* = (\text{Ker } h)^* \subset \text{Ker } g$, which implies $h(I^*) = 0$. Thus $N = K$, which is a contradiction.

(2) $\Rightarrow$ (1): Let $S \in \mathcal{T}_\tau$ be a simple R-module and $R \neq I \in \mathcal{L}_\tau$ be finitely generated with a nonzero homomorphism $f : I \to S$. Let $\text{Ker } f = K$, then we have $I/K \cong S \in \mathcal{T}_\tau$ and $R/I \in \mathcal{T}_\tau$ in the exact sequence $0 \to I/K \to R/K \to R/I \to 0$. Since $\mathcal{T}_\tau$ is closed under extensions, we get $R/K \in \mathcal{T}_\tau$. Hence $K \in \mathcal{L}_\tau$ and so $I^* \neq K^*$ by the hypothesis. Since $I \neq K$, it follows that there exists a maximal ideal J of R with $J \supseteq K$ but $J \not\supseteq I$. Hence $J + I = R$ and $J \cap I = K$, so

$$R/J = (J+I)/J \cong I/(I \cap J) = I/K \cong S.$$

Thus f can be extended to a homomorphism from R to S. Hence $\text{Ext}^1_R(R/I, S) = 0$. This completes the proof.

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