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ON CERTAIN REAL HYPERSURFACES OF A COMPLEX SPACE FORM

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Introduction

A complex n-dimensional Kaehler manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $M_n(c)$. The complete and simply connected complex space form consists of a complex projective space P_nC, a complex Euclidean space C^n or a complex hyperbolic space H_nC, according as $c > 0$, $c = 0$ or $c < 0$.

Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. Then it is seen that M has an almost contact metric structure (ϕ, ξ, η, g) induced from the Riemannian metric and the complex structure J of $M_n(c)$.

In the study of real hypersurfaces of a complex projective space P_nC, Takagi [13] classified all homogeneous real hypersurfaces of P_nC and showed that they are realized as the tubes of constant radius over Kaehler submanifolds if the structure vector ξ is principal. Namely, he proved the following

THEOREM A. *Let M be a connected real hypersurface of P_nC. Then M has constant principal curvatures and ξ is principal if and only if M is locally congruent to one of the following*

(A_1) a tube over a hyperplane CP^{n-1},

(A_2) a tube a totally geodesic $CP^k (1 \leqq k \leqq n-2)$,

(B) a tube over a complex quadric Q_{n-1},

(C) a tube over $CP^1 \times CP^{(n-1)/2}$ and $n (\geqq 5)$ is odd,

(D) a tube a complex Grassmann $G_{2,5}(C)$ and $n = 9$,

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(E) a tube a Hermitian symmetric space $SO(10)/U(5)$, *and* $n = 15$.

Acoording to Takagi's classification [13] the principal curvatures and their multiplicities of the above homogeneous real hypersurfaces are given.

On the other hand, real hypersurfaces of a complex hyperbolic space H_nC have also been investigated by Berndt [1], Montiel and Romero [10] and so on. Berndt [1] classified also homogeneous real hypersurfaces of H_nC and showed that they are realized as the tubes of arbitrary constant radius over Kaehler submanifolds. Namely, he proved the following

THEOREM B. *Let* M *be a connected real hypersurface of* H_nC $(n \geqq 2)$. *Then* M *has constant principal curvatures and* ξ *is principal if and only if* M *is locally congruent to one of the follwoing*

(A_1) *a horosphere in* H_nC,

(A_2) *a tube over* H_kC *for a* $k = 0, 1, \cdots, n-1$,

(B) *a tube over* H_nR.

For the principal curvatures and their multiplicities of the above hypersurfaces are also given in [1].

On the other hand, the linear transformation ϕ can be regarded as the operation for the Riemannian curvature tensor R on M. Maeda [8] studied about real hypersurface of P_nC with vanishing condition of this operation. Namely he proved the following

THEOREM C. *Let* M *be a real hypersurface of* P_nC. *If* $\phi R = 0$ *and if* ξ *is principal, then* M *is of type* A_1 *or type* A_2.

Let M be a real hypersurface of type A_1 or type A_2 in a complex projective space P_nC or that of type $A_0 \sim A_2$ in a complex hyperbolic space H_nC. Then M is said to be *of type A* for simplicity. By a theorem due to Okumura [11] and to Montiel and Romero [10] we have

THEOREM D. *If the shape operator* A *and the structure tensor* ϕ *commute to each other, then a real hypersurface of a complex space form* $M_n(c), c \neq 0$ *is locally congruent to be of type* A.

Recently, Kimura and Maeda [7] pointed out the importance of the normal distribution $\xi^{\perp}$ of the tangent bundle TM with respect to the structure vector ξ.

The purpose of the present paper to generalize Theorem C and Theorem D for the normal distribution $\xi^{\perp}$ in a real hypersurfce of a complex space form. We will prove the following

THEOREM 1. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geqq 3$. If the structure vector field ξ is principal and if there exists a tensor field ω of type (0, 3) such that*

$$\phi R|\xi^{\perp} = \omega \otimes \xi,$$

then M is of type A, where $\otimes$ denotes the tensor product.

1. Preliminaries

Let M be a real hypersurface of a complex n-dimensional complex space form $M_n(c)$ of constant holomorphic sectional curvature c, and let C be a unit normal vector field on a neighborhood of a point x in M. We denote by g the Riemannian metric tensor of M induced from that of $M_n(c)$ and we denote by $\bar{\nabla}$ and $\bar{\nabla}$ the Riemannian connection in $M_n(c)$ and in M, respectively. Then, by the Gauss formula, we have the relationship between ∇ and ∇ ; for any vector fields X and Y on M,

$$\bar{\nabla}_X Y = \nabla_X Y + g(AX, Y)C, \tag{1.1}$$

where A denotes the shape operator of M in $M_n(c)$ with to the unit normal C to M. Furthermore, we have another equation which is called the Weingarten formula ;

$$\bar{\nabla}_X C = -AX. \tag{1.2}$$

For any local vector field X on a neighborhood of x in M, the transfomations of X and C under the complex structure J in $M_n(c)$ can be given by

$$JX = \phi X + \eta(X)C, \; JC = -\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, where η and ξ denote a 1-form and a vector field on a neighborhood of x in M, respectively. Then it is seen that $g(\xi, X) = \eta(X)$. The set of tensors (ϕ, ξ, η, g) is called an *almost contact metric structure* on M. They satisfy the following

$$\phi^2 = -I + \eta \otimes \xi, \ \phi\xi = 0, \ \eta(\phi X) = 0, \ \eta(\xi) = 1, \tag{1.3}$$

where I denotes the identity transformation. Futhermore, the covariant derivatives of the structure tensors are given by

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \ \nabla_X \xi = \phi AX. \tag{1.4}$$

Since the ambient space is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are respectively given as follows :

(1.5)
$$\begin{aligned} R(X,Y)Z =& c\{g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &- 2g(\phi X, Y)\phi Z\}/4 + g(AY, Z)AX - g(AX, Z)AY, \end{aligned}$$

(1.6)
$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \xi(Y)\phi X - 2g(\phi X, Y)\xi\}/4,$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X.

2. Certain lemmas

Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geqq 3$. The linear transformation ϕ can be regarded as the oprator for the Riemannian curvature tensor R, that is, for any vector fields X, Y and Z we get

$$\begin{aligned} \phi R(X,Y,Z) = & \phi(R(X,Y)Z) - R(\phi X, Y)Z \\ & - R(X, \phi Y)Z - R(X,Y)(\phi Z). \end{aligned} \tag{2.1}$$

In this section we shall assume that the above operator satisfies

$$\phi R|\xi^{\perp} = \omega \otimes \xi \tag{2.2}$$

for any tensor field ω of type (0, 3), where $\otimes$ denotes the tensor product. In other words, we shall assume that

$$\phi R(X,Y,Z) = \omega(X,Y,Z)\xi \tag{2.3}$$

for any vector fields X, Y and Z orthogonal to ξ. Accordingly it implies that

$$g(\phi R(X,Y,Z),W) = 0 \tag{2.4}$$

for any vector fields orthogonal to ξ. Since the operator ϕ is skew-symmetric, by using the definition (2.1), the above equation (2.4) is rewritten by

$$\begin{aligned} g(R(\phi X,Y)Z,W) + g(R(X,\phi Y)Z,W) + g(R(X,Y)(\phi Z),W) \\ + g(R(X,Y)Z,\phi W) = 0. \end{aligned} \tag{2.4}$$

Taking account of the property $\phi^2 X = -X$, because X is orthogonal to ξ, and making use of Gauss equation (1.6), we can prove the following lemma by the direct calculation.

LEMMA 2.1. *There exists a tensor field ω of type (0,3) satisfies the condition (2.1) if and only if the shape operator A and the linear transformation ϕ satisfy*

(2.5)

$$\begin{aligned} &g(AX,Z)g((A\phi - \phi A)Y,W) + g(AY,W)g((A\phi - \phi A)X,Z) \\ &- g(AY,Z)g((A\phi - \phi A)X,W) - g(AX,W)g((A\phi - \phi A)Y,Z) = 0 \end{aligned}$$

for any vector fields X, Y, Z and W tangent to ξ.

REMARK 2.1. It is seen by Okumura [11] that the real hypersurface M of P_nC is of type A_1 or A_2 if and only if it satisfies $A\phi - \phi A = 0$ and it is also seen by Montiel and Romero [10] that the real hypersurface M of $M_n(c)$ is of type A_0, A_1 or A_2 if and only if it satisfies $A\phi - \phi A = 0$. Acoordingly, combining Lemma 2.1 and these properties we see that the real hypersurface of type A of $M_n(c)$ satisfies the condition (2.1).

REMARK 2.2. Let M be a ruled hypersurface of P_nC defined by Kimura [5]. Then on the ruled hypersurface, the structure vector field ξ is not principal. But it satisfies the equation (2.5). In fact, for the ruled hypersurface M of P_nC, there exists a unit vector U orthogonal to ξ which satisfies the following relationships ;

$$A\xi = a\xi + bU,\ AU = b\xi,\ AX = 0, \tag{2.6}$$

where X is a unit vector field orthogonal to ξ. Let V be a unit vector field defined by ϕU. Since V is also orthogonal to ξ and U because of the property of the almost contact metric structure, so is it, and moreover it satisfies $AV = 0$, $A\phi V = -b\xi$, which yields that $(A\phi - \phi A)U = 0$, $g(Y, U) = 0$ for any vector field Y orthogonal to ξ. This shows that left hand side of (2.5) vanishes identically, where W is identical with the vector field U. On the other hand, if W is orthogonal to ξ, U and $V = \phi U$, it is easily seen that $AW = A\phi W = 0$ and hence (2.5) is divided.

By this example the assumption in Theorem 1 that ξ is principal can not be rejected.

By the assumption of the dimension we can choose vector fields X and Y orthogonal to ξ such that X, ϕX and Y are orthonormal. Suppose that

$$AX = \lambda X,\ A\phi X = \mu\phi X,\ AY = \sigma Y. \tag{2.7}$$

LEMMA 2.2. *If M satisfies (2.2), then*

$$\begin{aligned} &\sigma(\lambda - \mu) = 0, \\ &(\lambda^2 + \mu^2)\{A\phi Y - \sigma\phi Y - g(\phi Y, A\xi)\xi\} = 0 \end{aligned} \tag{2.8}$$

for any vector fields satisfying (2.7).

Proof. By (2.7), it is clear that the equation (2.5) is equivalent to

$$\begin{aligned} &\lambda g(X, Z)g(A\phi Y - \sigma\phi Y, W) - \lambda g(X, W)g(A\phi Y - \sigma\phi Y, Z) \\ &\quad + \sigma(\lambda - \mu)\{g(\phi(X, W)g(Y, Z) - g(\phi X, Z)g(Y, W)\} = 0 \end{aligned} \tag{2.9}$$

for any vector fields Z and W orthogonal to ξ. We put $Z = \phi Y$ in (2.9). Because of $g(A\phi Y - \sigma\phi Y, \phi X) = \mu g(\phi Y, \phi X) - \sigma g(\phi Y, \phi X) = 0$, we get

$$\sigma(\lambda - \mu)\{g(\phi X, W)g(Y, \phi X) - g(Y, W)\} = 0,$$

which yields that the first equation of (2.8) holds. This means that we have

$$\lambda g(X, Z)g(A\phi Y - \sigma\phi Y, W) - \lambda g(X, W)g(A\phi Y - \sigma\phi Y, Z) = 0$$

by (2.9). When we put $W = A\phi Y - \sigma\phi Y - g(A\phi Y, \xi)\xi$ in this equation, we get

$$\lambda g(X, Z)\{|A\phi Y - \sigma\phi Y|^2 - g(A\phi Y, \xi)^2\} = 0$$

for any vector field Z orthogonal to ξ, because of $g(X, W) = 0$, where $|\cdot|$ denote the norm of vector fields on M. This shows that

$$\lambda\{|A\phi Y - \sigma\phi Y|^2 - (A\phi Y, \xi)^2\} = 0, (\text{that is})\ \lambda W = 0.$$

Similarly, if we can replace X with ϕX in the above discussion, we get $\mu W = 0$, which together with the above property implies that $(\lambda^2 + \mu^2)W = 0$. This completes the proof.

3. Proof of main theorem

In this section we shall prove Theorem 1. Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geqq 3$. We assume that the structure vector ξ is principal. The corresponding principal curvature is denoted by α, that is, we put

$$A\xi = \alpha\xi. \tag{3.1}$$

By a theorem by Maeda [9] in the case of $c > 0$ and a theorem by Ki and Suh [4] in the case of $c < 0$, the principal curvature α is constant on M. Then, by differentiation (3.1) convariantly and using (1.3) and so on, the followings equation is given ;

$$2A\phi A = \frac{c}{2}\phi + \alpha(A\phi + \phi A) \tag{3.2}$$

Let λ be any principal curvature on M whose coresponding principal vector X is orthogonal to ξ. In general, λ is the continuous function on M, and from (3.2) it follows that we obtain

$$(2\lambda - \alpha)A\phi X = (\alpha\lambda + \frac{c}{2})\phi X.$$

For any fixed principal curvature, let M_0 be a set consisting of points on M at which the value of λ is not equal to $\alpha/2$. It is trivial that the subset M_0 is open. We shall consider a point x which does not belong in the set M_0. Then we have $\lambda(x) = \alpha/2$, which together with (3.2) gives $\alpha^2 + c = 0$. This means that c is negative and

$$\lambda = \frac{\pm\sqrt{-c}}{2} \text{ on} M - M_0. \tag{3.3}$$

On the other hand, by the similar discussion, we get on M_0

$$A\phi X = \mu\phi X,\ \mu = \frac{\lambda\alpha + c/2}{2\lambda - \alpha} \tag{3.4}$$

for a principal vector field X with corresponding principal curvature λ. We can choose a unit principal field Y orthogonal to ξ so that X, ϕX and Y are mutually orthonormal by the assumption of the dimension. Let σ be the corresponding principal curvature. Then by Lemma 2.3, we get

$$\sigma(\lambda - \mu) = 0,\ (\lambda^2 + \mu^2)(A\phi Y - \sigma\phi Y) = 0. \tag{3.5}$$

Suppose here that there exists a point x in M_0 at which λ and μ are different. Then (3.5) yields $\sigma(x) = 0$ and $A\phi Y(x) = 0$. When we act the operator $A\phi A$ to the vector Y at x, the equation $c\phi Y(x) = 0$ is easily derived from the above equations. It implies that $c = 0$, a contradiction. This means that the principal curvature μ defined by (3.4) is always equal to the principal curvature λ on M_0, which is equivalent to the fact that the shape operator A and the linear transformation ϕ commute with each other and all principal curvatures on M_0 are constant. Moreover the number of distinct principal curvatures is just two, and they are roots of the quadratic equation $4y^2 - 4\alpha y - c = 0$, which shows that the root is different from $\alpha/2$. So, by the continuity of the principal curvatures, the set M_0 is identical with the hypersurface M itself or the set is empty.

In the latter case, the hypersurface M is of type A_0 in the complex hyperbolic space H_nC by the classification theorem due to Berndt [1], where as it is already seen that the other is of type A_1 or type A_2, inspite of the sign of the holomorphic curvature c.

As is already stated in Remark 2.2, the real hypersurface of type A in $M_n(c)$ satisfies the assumptions of Theorem 1. This completes the proof.

4. Real hypersurfaces of type A

In this section we shall concern about a characterization of a real hypersurfaces of type A of a complex space form $M_n(c)$, $c \neq 0$. As is generalized Theorem D, we prove

THEOREM 4.1. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. If there is a 1-form θ satisfying*

$$(A\phi - \phi A)|\xi^{\perp} = \theta \otimes \xi \tag{4.1}$$

and if ξ is principal, then M is locally congruent to the real hypersurface of type A of $M_n(c)$.

REMARK 4.1. (1) The real hypersurface of type A satisfies of course the condition (4.1).

(2) Let M be a ruled hypersurface of a complex projective space P_nC. By (2.6), we have $g((A\phi - \phi A)U, U) = 0$, $g((A\phi - \phi A)U, X) = 0$, $g((A\phi - \phi A)X, Y) = 0$ for any unit vectors X and Y orthogonal to ξ and U, which means that M satisfies the condition (4.1).

(3) If there is a 1-form θ satisfies

$$A\phi - \phi A = \theta \otimes \xi, \tag{4.2}$$

then it is easily seen that ξ is principal. Moreover we get $\theta = 0$. In fact, for any principal vector X corresponding to a principal curvature λ, we get

$$A\phi X = \phi AX + \theta(X)\xi,$$

from which it follows that we have $\theta(X) = 0$ by taking the inner product with ξ.

In order to prove Theorem 4.1 we shall show that the following property of M.

LEMMA 4.2. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. If it satisfies the condition*

$$g(\nabla_X A(Y), Z) = \delta g(AX, Y) g(Z, V), \tag{4.3}$$

where δ denotes the cyclic sum with respect to X, Y and Z orthogonal to ξ and V is the vector defined by $\nabla_\xi \xi$.

Proof. For any vector fields X, Y and Z orthogonal to ξ, the condition (4.1) implies that $g((A\phi - \phi A)(Y), Z) = 0$. Differentiating this equation covariantly in the direction of X, we get

$$\begin{aligned} &g(\nabla_X A(\phi A) + A \nabla_X \phi(Y) + A\phi(\nabla_X Y) - \nabla_X \phi(AY), Z) \\ &\quad - \phi \nabla_X A(Y) - \phi A(\nabla_X Y), Z) + g((A\phi - \phi A)(Y), \nabla_X Z) = 0. \end{aligned}$$

By taking account of (1.4) and the Codazzi equation (1.6), the above equation is reformed as

$$\begin{aligned} g(\nabla_X A(Y), \phi Z) &+ g(\nabla_X A(Z), \phi Y) \\ &= \eta(AY) g(X, AZ) + \eta(AZ) g(Y, AX) \\ &+ g(X, A\phi Y) g(Z, V) + g(X, A\phi Z) g(Y, V) \end{aligned} \tag{4.4}$$

In this equation we shall replace X, Y and Z cyclically and we shall then add the second equation to (4.4), from which substract the third one. By means of the Codazzi equation we get

$$\begin{aligned} 2g(\nabla_X A(Y), \phi Z) =& 2\eta(AZ) g(AX, Y) + g(X, V)\{g(Y, A\phi Z) - g(Z, A\phi Y)\} \\ &+ g(Y, V)\{g(X, A\phi Z) - g(Z, A\phi X)\}, \end{aligned}$$

which together with the condition (4.1) we can get the equation (4.3)

REMARK 4.2. A ruled hypersurface of P_nC satisfies the condition (4.3).

Proof of Thoerm 4.1. We can apply Lemma 4.2 to our situation and under an additional condition that ξ is principal it is seen that the shape operator is η-parallel, because of $V = \nabla_\xi \xi = 0$. By a theorem of Kimura and Maeda [7] and a theorem of Suh [12] it means that M is locally congruent one of real hypersurfaces of type A and type B of $M_n(c)$. But the case of type B cannot occur.

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