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DISTANCE SYSTEM

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1. Introduction

While the metrization problem as well as the pseudometrization problem was completely solved by Bing, Nagata and Smirov [3] in 1950, the quasimetrization problem was already considered and partially solved by Wilson [5] in 1951, and also by Norman [2] in 1965. A sufficient condition for the quasimetrization problem is given by Wilson and a counterexample which shows that the inverse of this quasimetrization theorem is not true is given by Norman. Besides, it is not so difficult to construct a space which can not be generated by the quasimetric [4]. Therefore a more generalized function of quasimetric, so called the linear distance function, is required [1], so that the class of spaces generated by this function should be larger than the class of spaces generated by the quasimetric. However, since this linear distance function has a chain structure as a range, every point of a topological space must have a neighborhood base as a chain, which is not a property that every spaces posess. In this paper, motivated by this property of range, we introduce a mathematical system, namely a distance functions (metric, pseudometric and quasimetric, e.t.c). The main idea of this system is that we are considering as a range of a function not only the linearly ordered set like $\mathbb{R}$, but also the partially ordered set containing a structure of lattice as a range of a function. The crucial difference of this function from the usual distance function is that ε could be chosen in the lattice structure and the image of the function could be the whole given partially ordered set. On the other hand this system is very closely related to the uniform space, in particular to the quasiuniform space.

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DEFINITION 2.1. Let (W,$\leq$) be an ordered set. A subset P of W is called a ***positive area*** of (W,$\leq$), if P$\neq \emptyset$ and for all $\epsilon, \delta \in$P there exits $\sigma \in$P such that $\sigma \leq \epsilon, \delta$.

(W,$\leq$,P) is called a ***range area***

Let M be a set and d a function from $M \times M$ into W. Let $(W, \leq, P)$ be a range area. Then α:=$(W, \leq, P, d)$ is called a ***distance system for M*** and d is called a ***distance function for M with the range area*** $(W, \leq, P)$.

For every x $\in M$ and $\varepsilon \in P$ the set $K_\varepsilon^\alpha(x) := \{x | x \in M, d(x,y) \leq \varepsilon\}$ is called the ***ε-sphere of x relative to α*** (If clear which α is meant, we write it briefly $K_\varepsilon(x)$).

A subset T of M is called *α-open*, if for each $x \in T$ there exists an $\varepsilon \in$ P with $K_\varepsilon^\alpha(x) \subseteq T$.

The following Theorem is trivial.

THEOREM 2.2. *Let $(W, \leq, P)$ be a range area and let $\alpha := (W, \leq , P, d)$ be a distance system for a set M. Then the following hold.*

(a) For all $x \in M$ and $\varepsilon, \delta \in P$ $\epsilon \leq \delta$ implies $K_\varepsilon^\alpha(x) \subseteq K_\delta^\alpha(x)$.

(b) For all $x \in M$ and $\varepsilon, \delta \in P$ there exists a $\sigma \in P$ such that $K_\sigma^\alpha(x) \subseteq K_\varepsilon^\alpha(x) \bigcap K_\delta^\alpha(x)$.

*(c) The set $\mathcal{O}_\alpha := \{T | T \subseteq M,\ T \text{ is } \alpha - open\}$ is a topology on M and it is said to be **generated** by α.*

DEFINITION 2.3. *Two distance systems $\alpha, \alpha'$are called **equivalent**, if $\mathcal{O}_\alpha = \mathcal{O}_{\alpha'}$.*

DEFINITION 2.4. Let (W,$\leq$,P) be a range area, M a set and let d be a distance function for M with the range area (W,$\leq$,P). Then

α:=(W,$\leq$,P,d) is called a *topological distance system for* M, if for all $x \in M$ and $\varepsilon \in P$ $K^{\alpha}_{\epsilon}(x)$ is the neighborhood of x relative to $\mathcal{O}_{\alpha}$.

We show next that every topology can be generated by a topological distance system.

NOTATION. For a set M of sets and an arbitrary x we set $M(x) := \{X | X \in M, x \in X\}$.

LEMMA 2.5. *(Criterion for a topology generated by a topological distance system) Let $(M, \mathcal{O})$ be a topological space. Let (W,$\leq$,P) be a given range and let d be a function of $M \times M$ in W. Then $\alpha := (W, \leq, P, d)$ is a topological distance system for M with $\mathcal{O} = \mathcal{O}_{\alpha}$ if and only if for all $x \in M$ the following holds.*

(1) For each $\varepsilon \in P$ there exists $U \in \mathcal{O}(x)$ such that $U \subseteq K_{\varepsilon}(x)$.

(2) For each $U \in \mathcal{O}(x)$ there exists $\varepsilon \in P$ such that $K_{\varepsilon}(x) \subseteq U$.

Proof. See [2].

DEFINITION 2.6. Let $(M, \mathcal{O})$ be a topological space. Let $\Delta := \{\mathcal{U} | \mathcal{U} \subseteq \mathcal{O}, \mathcal{U}$ is finite, $M \in \mathcal{U}\}$. For every $x \in M, \mathcal{U} \in \Delta$ we put $D^{\mathcal{U}}(x) := \bigcap_{U \in \mathcal{U}(x)} U$ and for every $\mathcal{U} \in \Delta$ we define a relation $R_{\mathcal{U}}$ on M as following: $xR_{\mathcal{U}}y :\leftrightarrow y \in D^{\mathcal{U}}(x)$ for all $x, y \in M$.

LEMMA 2.7. *Let $(M, \mathcal{O})$ be a topological space. Then the following hold.*

(a) For all $\mathcal{U}, \mathcal{V} \in \Delta$ $\mathcal{U} \cup \mathcal{V} \in \Delta$.

(b) For all $x \in M, \mathcal{U} \in \Delta$ $D^{\mathcal{U}}(x)$ is an open neighborhood of x.

(c) For all $\mathcal{U} \in \Delta$ we have $\iota \subseteq R_{\mathcal{U}}$, where ι is the identity relation on M.

(d) For all $x \in M$ and $\mathcal{U}, \mathcal{V} \in \Delta$ $\mathcal{U} \subseteq \mathcal{V}$ implies $D^{\mathcal{V}}(x) \subseteq D^{\mathcal{U}}(x)$.

(e) For all $\mathcal{U}, \mathcal{V} \in \Delta$ $R_{\mathcal{U} \cup \mathcal{V}} \subseteq R_{\mathcal{U}} \cap R_{\mathcal{V}}$.

Proof. We prove only (d) and (e)

(d) Let $\mathcal{U}, \mathcal{V} \in \Delta$ with $\mathcal{U} \subseteq \mathcal{V}$ and $x \in M$. Then obviously $\mathcal{U}(x) \subseteq \mathcal{V}(x)$ and also $\bigcap \mathcal{V}(x) \subseteq \bigcap \mathcal{U}(x)$, i.e. $D^{\mathcal{V}}(x) \subseteq D^{\mathcal{U}}(x)$.

(e) Let $\mathcal{U}, \mathcal{V} \in \Delta$. By (a) $\mathcal{U} \cup \mathcal{V} \in \Delta$. Let $x, y \in M$ with $xR_{\mathcal{U}\cup\mathcal{V}}y$. Then by (d) $y \in D^{\mathcal{U}\cup\mathcal{V}}(x) \subseteq D^{\mathcal{U}}(x), D^{\mathcal{V}}(x)$ and thus $xR_{\mathcal{U}}y, xR_{\mathcal{V}}y$. Therefore $R_{\mathcal{U}\cup\mathcal{V}} \subseteq R_{\mathcal{U}} \cap R_{\mathcal{V}}$.

THEOREM AND DEFINITION 2.8. *Let $(M, \mathcal{O})$ be a topological space and let Δ be defined as in 2.6. Let $W := \{S | \iota \subseteq S \subseteq M \times M\}$, $P := \{\varepsilon | \varepsilon \subseteq M \times M$, there is $\mathcal{U} \in \Delta$ with $R_{\mathcal{U}} \subseteq \varepsilon\}$, and $d : M \times M \to W, (x, y) \mapsto \iota \cup \{(x, y)\}$. We have then the following:*

(a) $\alpha := (W, \subseteq, P, d)$ is a distance system for M.

(b) For all $x \in M, \mathcal{U} \in \Delta$ $D^{\mathcal{U}}(x) = K_{R_{\mathcal{U}}}(x)$.

(c) α is a topological distance system and it generates $\mathcal{O}$.

We call the distance system α the quasiuniform distance system *of $(M, \mathcal{O})$.*

Proof. (a) It suffices to prove that P is a positive area of $(W, \subseteq)$. By 2.7 (c) $P \subseteq W$, and by $\Delta \neq \emptyset$ $P \neq \emptyset$. To show now is that for all $\varepsilon, \delta \in P$ there is $\sigma \in P$ with $\sigma \leq \varepsilon, \delta$. Let $\varepsilon, \delta \in P$. Then there are $\mathcal{U}, \mathcal{V} \in \Delta$ with $R_{\mathcal{U}} \subseteq \varepsilon, R_{\mathcal{V}} \subseteq \delta$. Let $\sigma := \varepsilon \cap \delta$. Then by 2.7 (e) $R_{\mathcal{U}\cup\mathcal{V}} \subseteq R_{\mathcal{U}} \cap R_{\mathcal{V}} \subseteq \varepsilon \cap \delta = \sigma$, hence $\sigma \in P$.

(b) Let $x \in M$ and $\mathcal{U} \in \Delta$. The following statements are equivalent. $z \in D^{\mathcal{U}}(x)$; $xR_{\mathcal{U}}z$; $\iota \cup \{(x, z)\} \subseteq R_{\mathcal{U}}$ (by 2.7 (c)); $d(x, z) \subseteq R_{\mathcal{U}}$; $z \in K_{R_{\mathcal{U}}}(x)$.

(c) We show the conditions (1),(2) of 2.5.

(1) Let $x \in M$ and $\varepsilon \in P$. There is $\mathcal{U} \in \Delta$ with $R_{\mathcal{U}} \subseteq \varepsilon$. Choose $U := D^{\mathcal{U}}(x)$. By 2.7 (a) $D^{\mathcal{U}}(x) \in \mathcal{O}(x)$. By (b) and 2.2 (a) $D^{\mathcal{U}}(x) = K_{R_{\mathcal{U}}}(x) \subseteq K_{\varepsilon}(x)$.

(2) Let $x \in M$ and $U \in \mathcal{O}(x)$. Choosing $\mathcal{U} := \{U, M\}, \mathcal{U} \in \Delta$. Moreover $D^{\mathcal{U}}(x) = U$. Let $\varepsilon := R_{\mathcal{U}}$. Then by (b) $K_{R_{\mathcal{U}}}(x) = D^{\mathcal{U}}(x) = U$.

In the next section we clarify the relationship between the distance system and the quasiuniform distance system

3. The Quasiuniform Distance System of a Topological Spaces

DEFINITION 3.1. Let M be a set. A subset $\mathcal{F}$ of $\mathcal{P}(M)$(:= power set of M) is called a *filter on M*, if the following conditions are satisfied.
(F1) $\mathcal{F} \neq \emptyset$, $\emptyset \notin \mathcal{F}$.
(F2) For all $U, V \in \mathcal{F}$ $\ U \cap V \in \mathcal{F}$
(F3) For all $T \subseteq M$, $U \in \mathcal{F}$ with $U \subseteq T$ $\ T \in \mathcal{F}$.

DEFINITION AND THEOREM 3.2. *Let M be a set and let $\mathcal{R}$ be a filter on $M \times M$. For $R \in \mathcal{R}$ and $x \in M$, we let $R(x) := \{z|(x,z) \in R\}$. A subset T of M is called $\mathcal{R}$-open, if to each $x \in T$ there exists $R \in \mathcal{R}$ with $R(x) \subseteq T$. Then the set $\mathcal{O}(\mathcal{R}) := \{T|T \subseteq M, T\ \mathcal{R}\text{-open}\}$ is a topology on M and it is said to be generated by $\mathcal{R}$.*

Proof. According to the theorem 2.2, let $W := \mathcal{P}(M \times M), P := \mathcal{R}$ and $\leq := \subseteq$. Then P is obviously a positive area of $(W, \subseteq)$. Let now $d : M \times M \to W, (x,y) \mapsto \{(x,y)\}$ and we define $\alpha := (W, \subseteq, P, d)$. We have only to show that for all $x \in M$ and $R \in \mathcal{R}$ $\ , K_R(x) = R(x)$. Let $x \in M$ and $R \in \mathcal{R}$. The following statements are equivalent. $z \in K_R(x)$; $d(x,z) \subseteq R$; $\{(x,z)\} \subseteq R$; $(x,z) \in R$; $z \in R(x)$. Thus $\mathcal{O}_\alpha = \mathcal{O}(\mathcal{R})$.

DEFINITION 3.3. *Let M be a set and A,B subsets of $\mathcal{P}(M \times M)$. The set $A \cdot B := \{(x,y)|$ There is $z \in M$ with $(x,z) \in A, (z,y) \in B\}$ is called the product relation of A,B.*

DEFINITION 3.4. Let M be a set. A subset $\mathcal{R}$ of $\mathcal{P}(M \times M)$ is called *quasiuniform structure on M*, if the following hold.
(U1) $\mathcal{R}$ is a filter on $M \times M$.
(U2) For all $S \in \mathcal{R}$ $\ \iota \subseteq S$.

(U3) For all $S \in \mathcal{R}$ there is $Q \in \mathcal{R}$ such that $Q \cdot Q \subseteq S$.

$(M, \mathcal{R})$ is called *quasiuniform space.*

A topological space $(M, \mathcal{O})$ is said to be ***generated by a quasiuniform space***, if there is a quasiuniform space $\mathcal{R}$ on M such that $\mathcal{O} = \mathcal{O}(\mathcal{R})$.

The relationship between the quasiuniform structure and the quasiuniform distance system is given by the following theorem.

THEOREM 3.5. *Let $(M, \mathcal{O})$ be a topological space with $M \neq \emptyset, \alpha := (W, \subseteq, P, d)$ be a quasiuniform distance system of $(M, \mathcal{O})$ defined in 2.8. Then P is a quasiuniform structure on M and $\mathcal{O}(P) = \mathcal{O}_\alpha = \mathcal{O}$.*

In particular ,every topological space is therefore generated by a quasiuniform structure.

Proof. We claim that (a) P is a quasiuniform structure on M, and (b) $\mathcal{O}(P) = \mathcal{O}_\alpha$.

(a) Since the properties (U1),(U2) are clearly satisfied, we show only the property (U3) of P. Let $\varepsilon \in P$. There is one $\mathcal{U} \in \Delta$ with $R_\mathcal{U} \subseteq \varepsilon$. We choose $\delta := R_\mathcal{U}$. It suffices to show $\delta = \delta \cdot \delta$. "$\subseteq$" is clear, since $\iota \subseteq \delta$. " $\supseteq$ " : Let $(x, y) \in \delta \cdot \delta$. There is $z \in M$ with $(x, z), (z, y) \in \delta$, i.e. $z \in D^\mathcal{U}(x), y \in D^\mathcal{U}(z)$. For all $V \in \mathcal{U}(x)$ $z \in D^\mathcal{U}(x)$ implies $z \in V$, thus $\mathcal{U}(x) \subseteq \mathcal{U}(z)$. With $y \in D^\mathcal{U}(z)$ it follows that $y \in D^\mathcal{U}(x)$. Therefore $xR_\mathcal{U}y$, i.e. $(x, y) \in R_\mathcal{U} = \delta$.

(b) For all $x \in M$ and $\varepsilon \in P$ $\varepsilon(x) = K_\varepsilon(x)$, because for any arbitrary $x \in M$ the followings are equivalent. $z \in \varepsilon(x)$; $(x, z) \in \varepsilon$; $\iota \cup \{(x, z)\} \subseteq \varepsilon$ (since $\iota \subseteq \varepsilon$); $d(x, z) \subseteq \varepsilon$; $z \in K_\varepsilon(x)$. Hence $\mathcal{O}(P) = \mathcal{O}_\alpha$.

We establish the next analogy between the quasiuniform distance system and the quasimetric distance system.

Let $(W, \subseteq, P, d)$ be a quasiuniform distance system of a topological space $(M, \mathcal{O})$. Suppose that d' is a quasimetric on M. Then $(\mathbb{R}^+ \cup \{0\}, \leq, \mathbb{R}^+, d')$ is a quasimetric distance system.

$(\mathbb{R}^+ \cup \{0\}, \leq, \mathbb{R}^+, d')$	$(W, \leq, P, d)$
$(\mathbb{R}^+ \cup \{0\}, \leq)$ an ordered set	$(W, \leq)$ an ordered set
0 is the smallest element of $\mathbb{R}^+ \cup \{0\}$	ι is the smallest element of W.
$+$ is associative connection on $\mathbb{R}^+ \cup \{0\}$ with the identity element 0	$\cdot$ is associative connection on W with identity element ι
$\leq$ is compatible relative to $+$	$\subseteq$ is compatible relative to $\cdot$.
$d'(x,x) = 0$ for all $x \in M$	$d(x,x) = \iota$ for all $x \in M$
$d'(x,z) \leq d'(x,y) + d'(y,z)$ for all $x, y, z \in M$	$d(x,z) \subseteq d(x,y) \cdot d(y,z)$ for all $x, y, z \in M$.

In the following we show that the concept of continuity in topological sense is equivalent to uniform continuity in quasiuniform distance systemic sense.

DEFINITION 3.6. Let $(M, \mathcal{O})$ be a topological space and $\alpha := (W, \subseteq, P, d)$ a quasiuniform distance system of $(M, \mathcal{O})$. $(M, \mathcal{O})$ is called *totally bounded*, if to every $\varepsilon \in P$ there is a finite subset E of M such that $\bigcup_{x \in E} K_\varepsilon(x) = M$.

THEOREM 3.7. *Every topological space is totally bounded.*

Proof. Let $(M, \mathcal{O})$ be a topological space. Let $\varepsilon \in P$. There is $\mathcal{U} \in \Delta$ with $R_\mathcal{U} \subseteq \varepsilon$. Obviously $\Omega := \{\mathcal{U}(x) | x \in M\} \subseteq \mathcal{P}(\mathcal{U})$. Since $\mathcal{U}$ is finite, Ω is finite. Let $\mathcal{D} := \{D^\mathcal{U}(x) | x \in M\}$. Then the function $\Omega \to \mathcal{D}$ which assigns $\mathcal{U} \mapsto \bigcap \mathcal{U}(x)$ is surjective, hence $\mathcal{D}$ is finite. In other words there is a finite subset E of M with $\{D^\mathcal{U}(x) | x \in E\} = \{D^\mathcal{U}(x) | x \in M\}$. We show that $\bigcup_{x \in E} K_\varepsilon(x) = M$. "$\subseteq$": Let $z \in M$. Then there is $x \in E$ such that $D^\mathcal{U}(z) = D^\mathcal{U}(x)$. By 2.7 (b) $z \in D^\mathcal{U}(z)$; hence $z \in D^\mathcal{U}(x)$. By 2.8 (b) and 2.2 (a) $D^\mathcal{U}(x) = K_{R_\mathcal{U}}(x) \subseteq K_\varepsilon(x)$; therefore $z \in K_\varepsilon(x)$. The another inclusion is trivial.

DEFINITION 3.8. Let $\alpha := (W, \leq, P, d), \alpha' := (W', \leq', P', d')$ be distance systems for M,M' respectively. A function $f : M \to M'$ is said to be ε-δ-continuous in $x \in M$ relative to α, α' respectively, if for every $\varepsilon \in P'$ there is $\delta \in P$, so that $d(x,z) \leq \delta$ implies $d'(f(x), f(z)) \leq' \varepsilon$ for all $z \in M$.

f is said to be *ε-δ-continuous relative to α, α'*, if f is *ε-δ-continuous at all $x \in M$*

f is said to be *uniformly continuous relative to α, α'*, if to each $\varepsilon \in P'$ there is $\delta \in P$ so that $d(x,y) \leq \delta$ implies $d'(f(x), f(y)) \leq' \varepsilon$ for all $x, y \in M$.

The following Theorem holds immediately.

THEOREM 3.9. *Let α, α' be distance systems for a set M,M' respectively, and let $f : M \to M'$ be a function. Then the followings hold.*

(a) f is ε-δ-continuous at $x \in M$ relative to α, α' if and only if f is continuous at $x \in M$ relative to topologies $\mathcal{O}_\alpha, \mathcal{O}_{\alpha'}$.

(b) f is ε-δ-continuous relative to α, α' if and only if f is continuous relative to the topologies $\mathcal{O}_\alpha, \mathcal{O}_{\alpha'}$.

THEOREM 3.10. *Let $\alpha := (W, \subseteq, P, d), \alpha' := (W', \subseteq, P', d')$ be the quasiuniform distance systems of topological spaces $(M, \mathcal{O})$, $(M', \mathcal{O}')$ respectively. Let $f : M \to M'$ be a function. Then f is continuous relative to $O_\alpha, O_{\alpha'}$ if and only if f is uniformly continuous relative to α, α'.*

Proof. " $\Longleftarrow$ " Let f be uniformly continuous relative to α, α'. Hence f is by 3.9 (b) continuous relative to $O_\alpha, O_{\alpha'}$.

" $\Longrightarrow$ " Let f be continuous relative to $O_\alpha, O_{\alpha'}$. Let Δ, Δ' be defined as in the definition 2.6. Let $\varepsilon \in P'$ be given. Then there is $\mathcal{U}' \in \Delta'$ with $R_{\mathcal{U}'} \subseteq \varepsilon$. Let $\mathcal{U} := \{f^{-1}(V') | V' \in \mathcal{U}'\}$. Since f is continuous, $\mathcal{U} \subseteq O$, hence $\mathcal{U} \in \Delta$. Choose $\delta := R_U$. Therefore $\delta \in P$. We show that $d(x,y) \subseteq \delta$ implies $d'(f(x), f(y)) \subseteq \varepsilon$ for all $x, y \in M$. Let $x, y \in M$ with $d(x,y) \subseteq R_{\mathcal{U}}$. Then $\iota \cup \{(x,y)\} \subseteq R_{\mathcal{U}}$, and thus $(x,y) \in R_{\mathcal{U}}$, i.e. $xR_{\mathcal{U}}y$. Hence $y \in D^{\mathcal{U}}(x) = \bigcap\{V | V \in \mathcal{U}, x \in$

$V\}$. Now $f(D^{\mathcal{U}}(x)) \subseteq \bigcap\{f(V)|V \in \mathcal{U}(x)\}$. Since for all $V \in \mathcal{U}$ there is $V' \in \mathcal{U}'$ with $f^{-1}(V') = V$ and $f(f^{-1}(V')) \subseteq V'$, we obtain $\bigcap\{f(V)|V \in \mathcal{U}, x \in V\} \subseteq \bigcap\{V'|V' \in \mathcal{U}', f(x) \in V'\} = D^{\mathcal{U}'}(f(x))$. Hence $f(D^{\mathcal{U}}(x)) \subseteq D^{\mathcal{U}'}(f(x))$. By $f(y) \in f(D^{\mathcal{U}}(x))$ $f(y) \in D^{\mathcal{U}'}(f(x))$. Therefore $f(x)R_{\mathcal{U}'}f(y)$ and hence $d'(f(x), f(y)) = \iota' \cup \{(f(x), f(y)\} \subseteq R_{\mathcal{U}'} \subseteq \varepsilon$.

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