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ON THE JOINT WEYL SPECTRUM WITH WEIGHT

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In [1,2], Chō studied the joint Weyl spectrum for a commuting pair of operators (bounded linear transformations). In this paper, we shall define a more general concept of the so called "weighted (joint) Weyl's spectrum" of a commuting pair of operators and study some of its properties.

Throughout the paper, H is a fixed(complex) Hilbert space of dimension $h \geq \aleph_0$, the cardinality of the set of natural numbers and $B(H)$ denotes the algebra of all bounded operators on H. For each cardinal $\alpha, \aleph_0 \leq \alpha \leq h$, let I_α denote the uniform closure of the two-sided ideal in $B(H)$ of all bounded operators of rank less than α. Then the I_α are precisely the proper closed two-sided ideals of $B(H)$. Of course, $I_{\aleph_0}$ is the ideal of compact operators and I_h is the maximal closed two sided ideal of $B(H)$. If $\aleph_0 \leq \alpha < \beta \leq h$, then $I_\alpha \subset I_\beta$ and $I_\alpha \neq I_\beta$[3]. For each operator T, $\widehat{T}$ denotes the coset $T + I_\alpha$ in the C^*-algebra $B(H)/I_\alpha$.

If T is α-compact i.e., $T \in I_\alpha$, then $\sigma(\widehat{T}) = \{0\}$, where $\sigma(\widehat{T})$ denotes the spectrum of $\widehat{T}$. Since I_α are self-adjoint ideals [3], $Re\ \sigma(\widehat{T}) = \{0\} = \sigma(\widehat{ReT})$.

In [4], Yadav and Arora defined the Weyl's spectrum of weight $\alpha, \omega_\alpha(T)$, of an operator T on H by

$$\omega_\alpha(T) = \bigcap_{K \in I_\alpha} \sigma(T + K).$$

For each operator T, $\omega_\alpha(T)$ is a nonempty compact subset of $\sigma(T)$, and $0 \notin \omega_\alpha(T)$ if and only if T is of the form $S + K$, where S is invertible and $K \in I_\alpha$[4].

THEOREM 1. *If $T = T_1 \oplus T_2$, then $\omega_\alpha(T) = \omega_\alpha(T_1) \cup \omega_\alpha(T_2)$.*

Proof. If $z \notin \omega_\alpha(T_1) \cup \omega_\alpha(T_2)$, then $T_1 - z = S_1 + K_1$, and $T_2 - z = S_2 + K_2$, where S_1, S_2 are invertible and $K_1, K_2 \in I_\alpha$. Thus $S_1 \oplus S_2$ is

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invertible on $H \oplus H$ and $K_1 \oplus K_2 \in I_\alpha$. Since $T - z = (S_1 \oplus S_2) + (K_1 \oplus K_2)$, $z \notin \omega_\alpha(T)$. Therefore $\omega_\alpha(T) \subseteq \omega_\alpha(T_1) \cup \omega_\alpha(T_2)$.

Conversely if $z \notin \omega_\alpha(T)$, then $T - z = S + K$ where S is invertible on $H \oplus H$ and K is $\alpha-$compact on $H \oplus H$. Hence there eixst invertible operators S_1 and S_2 and $\alpha-$compact operators K_1, K_2 such that

$$T_1 - z = S_1 + K_1 \text{ and } T_2 - z = S_2 + K_2.$$

Therefore $z \notin \omega_\alpha(T_1) \cup \omega_\alpha(T_2)$ and so $\omega_\alpha(T_1) \cup \omega_\alpha(T_2) \subseteq \omega_\alpha(T)$.

DEFINITION 2. Let $A = (A_1, A_2) \subset B(H)$ be a commuting pair of operators. The Taylor joint spectrum $\sigma^T(A)$ of A is defined by $\sigma^T(A) = \{z = (z_1, z_2) \in \mathbb{C}^2 : \alpha(A - z) \text{ is not invertible on } H \oplus H\}$, where

$$\alpha(A - z) = \begin{pmatrix} A_1 - z_1 & A_2 - z_2 \\ -(A_2 - z_2)^* & (A_1 - z_1)^* \end{pmatrix}.$$

DEFINITION 3. Let $A = (A_1, A_2) \subset B(H)$ be a commuting pair. The joint Weyl spectrum of weight α, $\omega_\alpha(A)$, of A is defined by

$$\omega_\alpha(A) = \cap\{\sigma^T(A+K) : K = (K_1, K_2) \subset I_\alpha \text{ and } A+K = (A_1+K_1, A_2+K_2)$$

is a commuting pair$\}$.

$z = (z_1, z_2)$ in $\mathbb{C}^2$ is said to be joint eigenvalue of $A = (A_1, A_2)$ if there exist a nonzero vector x such that $A_i x = z_i x (i = 1, 2)$. $\sigma_p(A)$ is the set of joint eigenvalues of A.

$z = (z_1, z_2)$ in $\mathbb{C}^2$ is said to be joint residual eigenvalue of $A = (A_1, A_2)$ if there exists a non-zero vector x such that $A_i^* x = \overline{z_i} x (i = 1, 2)$. $\sigma_r(A)$ is the set of joint residual eigenvalues of A.

From the self-adjointness of I_α we see that for any commuting pair $A = (A_1, A_2)$,

$$\omega_\alpha(A^*) = \overline{\omega_\alpha(A)} = \{(\bar{z}_1, \bar{z}_2) : z = (z_1, z_2) \in \omega_\alpha(A)\}.$$

REMARK. For any commuting pair $A = (A_1, A_2)$, $0 \in \omega_\alpha(A)$ iff there eixst a nonsigular pair $S = (S_1, S_2)$ on H and $K = (K_1, K_2) \subset I_\alpha$ such that $A = S + K$ and $A + K$ is a commuting pair.

THEOREM 4. *For any commuting pair* $A = (A_1, A_2)$, $\omega_\alpha(A)$ *is a nonempty compact subset of* $\sigma^T(A)$.

Proof. That $\omega_\alpha(A)$ is a compact subset of $\sigma^T(A)$ follows from the definition. We claim that $\sigma^T(\widehat{A}) \subset \omega_\alpha(A)$. Let $z = (z_1, z_2) \in \sigma^T(\widehat{A})$. Then $\widehat{A} - z$ is singular in $B(H)/I_\alpha$. Let $z \in \omega_\alpha(A)$. Then by Remark, there exist a nonsingular pair $S = (S_1, S_2)$ on H and $K = (K_1, K_2) \subset I_\alpha$ such that $A - z = S + K$ and $A + K$ is a commuting pair. Hence $\widehat{A} - z = \widehat{S}$, where $\widehat{S}$ is nonsingular in $B(H)/I_\alpha$. This is a contradiction. Hence $z \in \omega_\alpha(A)$ and therefore $\omega_\alpha(A)$ is a nonempty compact subset of $\sigma^T(A)$.

The following theorem is a generalization of Chō and Takaguchi's Theorem [1,2].

THEOREM 5. *For any commuting pair* $A = (A_1, A_2) \subset B(H)$, *of operators*

$$\sigma^T(A) - \omega_\alpha(A) \subseteq \sigma_p(A) \cup \sigma_r(A).$$

Proof. Let $z = (z_1, z_2) \in \sigma^T(A) - \omega_\alpha(A)$. Then since $z \notin \omega_\alpha(A)$, there exists $K = (K_1, K_2) \subset I_\alpha$ such that $z \notin \sigma^T(A + K)$ and $A + K$ is a commuting pair.

Therefore,

$$\alpha(A + K - z) = \begin{pmatrix} A_1 + K_1 - z_1 & A_2 + K_2 - z_2 \\ -(A_2 + K_2 - z_2)^* & (A_1 + K_1 - z_1)^* \end{pmatrix}$$

is invertible.

So $\alpha(A + K - z)^*$ is invertible. Let

$$T = \alpha(A + K - z)^{*^{-1}} \cdot \begin{pmatrix} K_1^* & -K_2 \\ K_2^* & K_1 \end{pmatrix}.$$

Then from the self-adjointness of I_α, $T \subseteq I_\alpha(H \oplus H)$ and $\alpha(A - z)^* = \alpha(A + K - z)^*(I - T)$. Since $T \subseteq I_\alpha(H \oplus H)$, the null space of $I - T$ is nonzero. So there exists a nonzero vector $x \oplus y \in H \oplus H$ such that

$$\alpha(A - z)^*(x \oplus y) = 0 \oplus 0.$$

And since

$$\alpha(A-z)\cdot\alpha(A-z)^* = \begin{pmatrix} \sum_{i=1}^{2}(A_i - z_i)(A_i - z_i)^* & 0 \\ 0 & \sum_{i=1}^{2}(A_i - z_i)^*(A_i - z_i) \end{pmatrix},$$

we get that $z \in \sigma_r(A)$ if x is nonzero, and we get that $z \in \sigma_p(A)$ if y is nonzero. Thus the proof is complete.

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