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REORIENTATIONS OF TOWERS

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Let P be a finite ordered set. The *covering graph* $C(P)$ of P is the graph with vertices its elements and with edges joining two comparable elements x and y whenever either x covers $y(x \succ\!- y)$ or y covers $x(y \succ\!- x)$, that is, there is no element strictly between x and y. There may be several ordered sets with the same covering graph as P. Each of their diagrams is said to be a *reorientation* of (the diagram of) P. Then we say that a property ρ about ordered sets is a *diagram invariant* if, for every ordered set P which satisfies ρ, any reorientation of P also satisfies ρ.

By the *ordinal sum* P of ordered sets $P_1, \cdots, P_n$, we mean the set $P_1 \cup \cdots \cup P_n$ with the orders of $P_1, \cdots, P_n$ and the new relations $x < y$ whenever $x \in P_i$, $y \in P_j$ and $i < j$, and we write $P = P_1 \oplus \cdots \oplus P_n$. A *tower* is defined to be an ordinal sum of antichains.

In this note we consider some aspects of reorientations of towers. We assume throughout that all ordered sets are finite.

First we have a nontrivial diagram invariant (cf. [2], [3]). Here we need some notations. For an ordered set P, we denote by min P and max P the sets of all minimal elements and all maximal elements, respectively, of P.

THEOREM 1. *Let P and Q be ordered sets such that $C(P) = C(Q)$. If P is a tower of length at most 2, then so is Q.*

Proof. When P is an antichain, it is trivial. Suppose that $P = A \oplus B$, where A and B are antichains. If $A' = A \cap \min Q \neq \phi$, then $b \succ\!- a$ for any $a \in A'$ and $b \in B$, Hence $Q = A' \oplus B \oplus (A - A')$. Similarly, if $B' = B \cap \min Q \neq \phi$, then $Q = B' \oplus A \oplus (B - B')$. Finally suppose that $P = A \oplus B \oplus C$, where A, B and C are antichains. If $P' = B \oplus (A \cup C)$ such that $A \cup C$ is an antichain, then $C(Q) = C(P) = C(P')$. Now we can apply the preceding argument again.

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Next, we consider the dimension of reorientations of a tower of arbitrary length.

LEMMA 2. *Let P and Q be ordered sets such that $C(P) = C(Q)$. If $P = A_1 \oplus \cdots \oplus A_n$ for antichains $A_1, \cdots, A_n$, then in Q each A_i is either an antichain or the disjoint union of two antichains A'_i and A'' such that $A'_i \subseteq \max Q$ and $A''_i \subseteq \min Q$.*

Proof. If A_i is not an antichain in Q, then $a > b$ in Q for some a and b in A_i. Let $A'_i = \{x \in A_i | x > b\}$ and $A''_i = \{x \in A_i | x < a\}$. If $y \succ -x$ in Q for some $x \in A'_i$, then there is an edge betwen y and b, which is a contradiction. Hence, $A'_i \subseteq \max Q$. Similarly, $A''_i \subseteq \min Q$. Therefore A'_i and A''_i are antichains and their union is A_i.

THEOREM 3. *Let P and Q be ordered sets such that $C(P) = C(Q)$. If $P = A_1 \oplus \cdots \oplus A_n$ for antichains $A_1, \cdots, A_n$, then Q has dimension at most 2.*

Proof. To prove that Q has dimension at most 2, we shall show by induction on n that Q can be embedded in the direct product of two chains C and D in the following way :

i) According to the types of antichains in Lemma 2, we write
$\pi_C(A_i) = [c(i,1), c(i,2)]$, $\pi_D(A_i) = [d(i,1), d(i,2)]$,
$\pi_C(A'_i) = [c'(i,1), c'(i,2)]$, $\pi_D(A'_i) = [d'(i,1), d'(i,2)]$,
$\pi_C(A''_i) = [c''(i,1), c''(i,2)]$, $\pi_D(A''_i) = [d''(i,1), d''(i,2)]$,
for $i = 1, \cdots, n$, where π_T denotes the projection on $T \in \{C, D\}$.

ii) If X and Y are two distinct antichains of the types in Lemma 2 then $\pi_T(X) \cap \pi_T(Y) = \phi$ for $T \in \{C, D\}$.

iii) Locate none of $A_1, \cdots, A_{k-1}$ on the left side of A_k in the embedding, i.e., according to the types of antichains in Lemma 2,

$$([c(k,2), 1] \times [0, d(k,1)]) \cap A_i = \phi \text{ for } i = 1, \cdots, k-1,$$
$$([c'(k,2), 1] \times [0, d'(k,1)]) \cap A'_i = \phi \text{ for } i = 1, \cdots, k-1,$$
$$([c''(k,2), 1] \times [0, d''(k,1)]) \cap A''_i = \phi \text{ for } i = 1, \cdots, k-1,$$

where 0 and 1 denote as usual the least element and the largest element, respectively, of a chain.

Suppose that $Q - A_n$ is embedded in $C \times D$ as above. We now insert some intervals in C and D for A_n to be embedded in $C \times D$. To do this we have the following four cases to consider.

Case 1. $A_n \subseteq \min Q$. Embed A_n by inserting two intervals $[c(n,1), c(n,2)] = \pi_C(A_n)$ and $[d(n,1), d(n,2)] = \pi_D(A_n)$ such that $c(n-1,1) \succ c(n,2)$ and $d(n,1) = 0$.

Case 2. $A_n \subseteq \max Q$. This is the dual of Case 2.

Case 3. $A_n = A'_n \cup A''_n$, where $\phi \neq A'_n \subseteq \min Q$ and $\phi \neq A'' \subseteq \min Q$. In this case, A_{n-2} is also divided into two antichains A'_{n-2} and A''_{n-2} as in Lemma 2. Now embed A'_n by inserting two intervals $[c'(n,1), c'(n,2)] = \pi_C(A'_n)$ and $[d'(n,1), d'(n,2)] = \pi_D(A'_n)$ such that $c'(n,1) \succ c'(n-2,2)$ and $d'(n,1) \succ d(n-1,2)$ and embed A''_n dually.

Case 4. $A_n \cap \min Q = \phi = A_n \cap \max Q$. In this case, A_{n-1} is divided into two antichains ${A_{n-1}}'$ and ${A_{n-1}}''$ as in Lemma 2. Now embed A_n by inserting two intervals $[c(n,1), c(n,2)] = \pi_C(A_n)$ and $[d(n,1), d(n,2)] = \pi_D(A_n)$ such that $c'(n-1,1) \succ c(n,2)$ and $d(n,1) \succ d''(n-1,2)$.

COROLLARY 4. *Every reorientation of a chain has dimension at most 2.*

Towers are quite special ordered sets. We not yet found other classes of ordered sets of dimension 2 whose reorientations have dimension at most 2. In fact, there are ordered sets of dimension 2 which have reorientations of arbitrarily large dimension. For example, see the ordered set P_n of dimension n in Section 2 [1] from which we obtain a planar lattice by rotating the diagram 90 degree in either direction.

References

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