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ON THE ACTION OF HECKE OPERATORS ON THE DRINFEL'D CUSP FORMS OF SMALL WEIGHTS

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0. Introduction

In the classical theory of modular forms for $SL(2,\mathbf{Z})$ there exists a basis consisting of eigenforms of Hecke algebra for the space of cusp forms of given weight, and their eigenvalues are real algebraic numbers([6], [7]). The proof uses Petersson inner product. But in the Drinfel'd modular theory we still do not have the analogous statements. To have something like Petersson inner product we have to develop integration theory for function field of positive characteristic, which is not settled yet.

In this note, however, we will calculate the action of Hecke operators for the ideals of degree 1 or 2 on the Drinfel'd cusp forms of weight $2q^2-2$ which is the next simplest case other than those of weight q^2-1 and for ideals of degree 1 on those of weight $3q^2-3$. Then we obtain common eigenforms of the Hecke operators for the ideals of degree 1 or 2, and see that their eigenvalues are real algebraic. These would give some evidence for the existence of the statement analogous to the classical theory.

1. Notations, Definitions and Basic Properties

Let $K=\mathbf{F}_q(T)$ be the rational function field over a finite field $\mathbf{F}_q$ and $A=\mathbf{F}_q[T]$ be its ring of integers. We assume that $\operatorname{char} K>3$ for simplicity. Denote by $K_\infty=\mathbf{F}_q((T))$ the completion of K at ∞ and by C the completion of algebraic closure of K_∞.

DEFINITION 1.1. An element $z\in C$ is said to be *real* if $z\in K_\infty$.

In this note we always mean by a Drinfel'd module the Drinfel'd module of rank 2 over C on A unless otherwise stated. It is known

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that the set of Drinfel'd modules is parametrized by $\Omega = C - K_\infty$ and $GL(2, A)$ acts on Ω as linear fractional transformations. Since Ω can be given the structure of rigid analytic space, we have the notion of holomorphicity and so we can consider modular forms for $GL(2, A)$.

A Drinfel'd module ϕ is given by

$$\phi_T = T + \bar{g}\tau + \overline{\Delta}\tau^2$$

where τ maps z to z^q. Then it is known that $\bar{g}$(resp. $\overline{\Delta}$) is a modular form of weight $q-1$ (resp. q^2-1), and the algebra of modular forms is the algebra generated by $\bar{g}$ and $\overline{\Delta}$ over C.

Let $L = \bar{\pi}A$ be the rank 1 lattice associated to the Carlitz module ρ given by $\rho_T = T + \tau$.

Let $t(z) = e_L^{-1}(\bar{\pi}z)$ where $e_L(z) = z\prod'_{\lambda \in L}\left(1 - \frac{z}{\lambda}\right)$. Then modular forms have t-expansions. For all of these we refer to [1], [2], or [3].

Let $\mathfrak{a}$ be an ideal of A. Then we can define the Hecke operator $T_\mathfrak{a}$ on the set M_k of modular forms of weight k. In fact, $T_{\mathfrak{a}\mathfrak{b}} = T_\mathfrak{a}T_\mathfrak{b}$ where $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of A. Therefore we need only consider $T_\mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal of A. In our case $T_\mathfrak{p}$ has a simple forms as follows :

For $f \in M_k$ and $\mathfrak{p} = (p)$, where p is a monic irreducible polynomial of degree d, then

$$T_\mathfrak{p}f(z) = p^k f(pz) + \sum_{\substack{b \in A \\ \deg b < d}} f((z+b)/p). \tag{1.2}$$

As in the classical case, $T_\mathfrak{p}f$ is cuspidal if f is.

Let Λ be an $\mathbf{F}_q$-lattice in C and $S_k = S_{k,\Lambda} = \sum_{\lambda \in \Lambda}(z+\lambda)^{-k}$. Then we have

PROPOSITION 1.3. ([2], (3.4)) *Let Λ be an $\mathbf{F}_q$-lattice in C. Then there exists a polynomial $G_k = G_{k,\Lambda}$, called the k-th Goss polynomial of Λ, with the following properties*

(i) $S_k = G_k(t)$ *where* $t = t_\Lambda = S_{1,\Lambda}$
(ii) $G_k(X) = X(G_{k-1}(X) + \alpha_1 G_{k-q}(X) + \cdots + \alpha_i G_{k-q^i} + \cdots)$, $k - q^i \geq 0$
(iii) G_k *is monic of degree* k
(iv) $G_k(0) = 0$

(v) *If* $k \le q$, *then* $G_k(X) = X^k$
(vi) $G_{pk} = (G_k)^p$ *where* $p = \mathrm{char}(\mathbf{F}_q)$
(vii) $X^2 G_k'(X) = kG_{k+1}(X)$.

Here α_i's are given by $e_\Lambda(z) = \sum_{i \ge 0} \alpha_i z^{q^i}$.

We can express the action of $T_\mathfrak{p}$ on the t-expansions of a modular form of weight k using Goss polynomials as follows (see [2], (7.3)) ;

Let $\Lambda_\mathfrak{p} = \operatorname{Ker} \rho_p$, and $G_{i,\mathfrak{p}}$ the i-th Goss polynomial of $\Lambda_\mathfrak{p}$. Then

$$T_\mathfrak{p}\left(\sum a_i t^i\right) = p^k \sum a_i t_p^i + \sum a_i G_{i,\mathfrak{p}}(pt), \tag{1.4}$$

where $t_p(z) = t(pz) = t^{|p|}/f_p(t)$. Here f_p is defined by

$$f_p(X) = \rho_p(X^{-1})X^{|a|}.$$

Our main interest in this note is the set of cusp forms of weight $2q^2-2$ (resp. $3q^2-3$), which is a 2-dimensional vector space with basis $\{g^{q+1}\Delta, \Delta^2\}$ (resp. a 3-dimensional vector space with basis $\{g^{2q+2}\Delta, g^{q+1}\Delta^2, \Delta^3\}$) ([2], [3]), where $g = \bar\pi^{1-q}\bar g$ and $\Delta = \bar\pi^{1-q^2}\overline{\Delta}$.

We need some informations about the coefficients of g and Δ in the t-expansions.

Proposition 1.5. ([2], (5.10)) (i) *The only powers of t divisible by $q-1$ occur in the t-expansions of g and Δ.*

(ii) *Let $\sum a_i s^i$ be the expansion of g or Δ, where $s = t^{q-1}$. Then $a_i \ne 0$ only if $i \equiv 0$ or $1 \bmod q$.*

The first few terms of g and Δ are

(1.6)
$$g = 1 - [1]s - [1]s^{q^2-q+1} + [1]s^{q^2} - [1]^2 s^{q^2+1} + \text{higher terms}$$

(1.7)
$$\begin{aligned}\Delta = -s + s^q - [1]s^{q+1} - s^{q^2-q+1} + s^{q^2} - ([1]-[1]^q)s^{q^2+1} \\ - [1]^q s^{q^2+q} + [1]^{q+1}s^{q^2+q+1} + \text{higher terms}\end{aligned}$$

where $[1] = T^q - T$.

2. Some Lemmas

LEMMA 2.1. *$G_{k(q-1)}$ is a polynomial in X^{q-1}.*

Proof. We will prove this by induction. Since $G_{q-1} = X^{q-1}$, we are done for $k = 1$. Assume that $G_{j(q-1)}$ is a polynomial in X^{q-1} for $j < k$. Then

$$\begin{aligned}
G_{k(p-1)} &= X\left(\sum_{i_1 \geq 0} \alpha_{i_1} G_{k(q-1)-q^{i_1}}\right) \\
&= X\left(\sum_{i_1 \geq 0} \alpha_{i_1} G_{m_{i_1}(q-1)-1}\right) \quad \text{where } m_{i_1} = k - \frac{q^{i_1}-1}{q-1} \\
&= X^2\left(\sum_{i_1 \geq 0} \sum_{i_2 \geq 0} \alpha_{i_1}\alpha_{i_2} G_{m_{i_1}(q-1)-q^{i_2}-1}\right) \\
&= X^2\left(\sum_{i_1 \geq 0} \sum_{i_2 \geq 0} \alpha_{i_1}\alpha_{i_2} G_{m_{i_1 i_2}(q-1)-2}\right) \text{ where } m_{i_1 i_2} = m_{i_1} - \frac{q^{i_2}-1}{q-1} \\
&\cdots \\
&= X^{q-1}\left(\sum_{i_1 \geq 0} \sum_{i_2 \geq 0} \cdots \sum_{i_{q-1} \geq 0} \alpha_{i_1} \cdots \alpha_{i_{q-1}} G_{m_{i_1 \cdots i_{q-1}}(q-1)-(q-1)}\right)
\end{aligned}$$

Since $m_{i_1} \cdots i_{q-1} < k$, we are done. □

Let $\sum a_i s^i$ be the s-expansion of $g^{q+1}\Delta$ or Δ^2 for the case of weight $2q^2 - 2$. Then by Proposition 1.5, $a_i \neq 0$ only if $i \equiv 0$, 1 or 2 mod$q$. By (1.6) and (1.7),

$$\begin{aligned}
g^{q+1}\Delta &= -s + [1]s^2 + 0(s^3) \qquad \text{and} \\
\Delta^2 &= s^2 + 0(s^3).
\end{aligned}$$

Hence to express $T_\mathfrak{p}(g^{q+1}\Delta)$ and $T_\mathfrak{p}(\Delta^2)$ as linear combinations of $g^{q+1}\Delta$ and Δ^2, we need only the coefficients of s and s^2 of $T_\mathfrak{p}\left(\sum a_i s^i\right)$. The first summand in the right hand side of (1.4) does not contribute. Therefore we only need the coefficients of X^{q-1} and X^{2q-2} in $G_{i(q-1)}$. Similarly we only need the coefficients of s, s^2 and s^3 of $T_\mathfrak{p}(\sum a_i s^i)$ for the case of weight $3q^2 - 3$.

LEMMA 2.2. *If $i \equiv 0 \bmod q$, $G_{i(q-1)}$ has no X^{q-1}, and X^{2q-2} terms.*

Proof. Let $i = rq$. Then

$$\begin{aligned} G_{i(q-1)}(X) &= G_{r(q-1)q}(X) \\ &= (G_{r(q-1)}(X))^q. \end{aligned}$$

Hence $G_{i(q-1)}(X)$ has no terms other than X^{mq}. □

LEMMA 2.3. *If $i \equiv 1 \bmod q$, $G_{i(q-1)}$ has no X^{q-1}, X^{2q-2} and X^{3q-3} terms except for $i = 1$.*

Proof. Let $i = rq + 1$ and $r > 0$. Then

$$G_{i(q-1)}(X) = G_{(rq+1-r)q-1}(X).$$

Hence by Proposition 1.2, (vii)

$$\begin{aligned} X^2 G'_{i(q-1)}(X) &= -G_{(rq+1-r)q}(X) \\ &= -(G_{rq+1-r}(X))^q. \end{aligned}$$

But it is easy to see that X^2 divides $G_m(X)$ for $m > 1$, so $G_{i(q-1)}(X)$ does not contain X^{q-1} and X^{2q-2} terms. Since $\operatorname{char} K > 3$, $G_{i(q-1)}(X)$ has no X^{3q-3} term. □

LEMMA 2.4. *Let $i = rq + 2$. Then*

(i) *the coefficient of X^{q-1} in $G_{i(q-1)}$ is*

$$\begin{array}{lll} -2\alpha_k & \text{if } r = \dfrac{q^{k-1}-1}{q-1}, & k \geq 1 \\ 0 & \text{otherwise} & \end{array}$$

(ii) *the coefficient of X^{2q-2} in $G_{i(q-1)}$ is*

$$\begin{array}{lll} \alpha_k^q & \text{if } r = \dfrac{q^{k}-1}{q-1}, & k \geq 0 \\ 0 & \text{otherwise} & \end{array}$$

(iii) *the coefficient of X^{3q-3} in $G_{i(q-1)}$ is 0.*

Proof. From ([2], (3.8))

$$G_m = \sum_{j \le m-1} \sum_{\underline{i}} \binom{j}{\underline{i}} \alpha^{\underline{i}} X^{j+1}$$

where $\underline{i} = (i_0, i_1, \ldots, i_s)$, $j = i_0 + i_1 + \cdots + i_s$, $m - 1 = i_0 + i_1 q + \cdots + i_s q^s$, $\alpha^{\underline{i}} = \alpha_0^{i_0} \alpha_1^{i_1} \cdots \alpha_s^{i_s}$ and $\binom{j}{\underline{i}} = j!/(i_0! \cdots i_s!)$.

Let $m = (rq+2)(q-1)$ and $j = q-2$. Then the coefficient X^{q-1} is

$$\sum_{\underline{i}} \binom{q-2}{\underline{i}} \alpha^{\underline{i}},$$

where

(1) $i_0 + i_1 + \cdots + i_s = q - 2$ and
(2) $i_0 + i_1 q + \cdots + i_s q^s = (rq + 2 - r)q - 3$.

From (2), $i_0 \equiv -3 \bmod q$. Therefore i_0 must be $q-3$. Hence $i_k = 1$ for a unique $k \ge 1$, and other i's are 0. Taking account of these we get $r = \frac{q^{k-1}-1}{q-1}$ and (i).

Now let $j = 2q-3$. Then again $i_0 \equiv -3 \bmod q$ and

$$i_0 + i_1 + \cdots + i_s = 2q - 3.$$

Hence $i_0 = q-3$ or $2q-3$. If $i_0 = 2q-3$, then other i's are 0 and so $r = 0$. Conversely if $r = 0$, then $i_0 = 2q-3$. So the coefficient of X^{2q-2} in G_{2q-2} is 1, which is α_0 by definition.

Assume $i_0 = q-3$. Then $i_1 + \cdots + i_s = q$. If $i_1, \ldots, i_s$ are less then q, then $(2q-3)!/(q-3)!\,i!\cdots i_s! = 0$ in C. Therefore $i_k = q$ for a unique k and other i's are 0. And in this case $r = \frac{q^k-1}{q-1}$, $k \ge 0$. But $\binom{2q-3}{(q-3,q)} \equiv i \bmod q$. Thus we get (ii).

Let $j = 3q-4$. As before $i_0 \equiv -3 \bmod q$ and so $i_0 = q-3$ or $2q-3$. If $i_0 = 2q-3$, then other i's are less than q. therefore $\binom{3q-4}{(2q-3, i_1, \ldots, i_s)} = 0$. If $i_0 = q-3$, then $\binom{eq-4}{(q-3, i_1, \ldots, i_s)} = \frac{(q-2)(q-1)q\cdots 2q \cdots (3q-4)}{i_0!\, i_2! \cdots i_s!}$. The numerator is divisible by q^2 but the denominator is divisible only by q if q is a prime, since $i_1 + \cdots + i_s = 2q-1$. The same is true for q a power of a prime. Therefore the coefficient of X^{3q-3} is 0 in this case also. □

LEMMA 2.5. *Let $i = rq + 3$. Then*

(i) *the coefficient of X^{q-1} in $G_{i(q-1)}$ is*

$$\begin{array}{lll} 3\alpha_k^2 & \text{if } r = \dfrac{2q^{k-1}-2}{q-1}, & k > 0 \\ 6\alpha_{k_1}\alpha_{k_2} & \text{if } r = \dfrac{q^{k_1-1}+q^{k_2-1}-2}{q-1}, & k_1 > k_2 > 0 \\ 0 & \text{otherwise} & \end{array}$$

(ii) *the coefficient of X^{2q-2} in $G_{i(q-1)}$ is*

$$\begin{array}{lll} -3\alpha_k^{q+1} & \text{if } r = \dfrac{(q+1)q^{k-1}-2}{q-1}, & k > 0 \\ -3\alpha_{k_1}\alpha_{k_2}^q & \text{if } r = \dfrac{q^{k_1}+q^{k_2-1}-2}{q-1}, & k_1 \neq k_2 > 0 \\ -3\alpha_k & \text{if } r = \dfrac{q^{k-1}-1}{q-1}, & k > 0 \\ 0 & \text{otherwise} & \end{array}$$

(iii) *the coefficient of X^{3q-3} in $G_{i(q-1)}$ is*

$$\begin{array}{lll} 1 & \text{if } r = 0 & \\ 2\alpha_{k_1}^q\alpha_{k_2}^q & \text{if } r = \dfrac{q^{k_1}+q^{k_2}-2}{q-1}, & k_1 \neq k_2 > 0 \\ 2\alpha_k^q & \text{if } r = \dfrac{q^k-1}{q-1}, & k > 0 \\ \alpha_k^{2q} & \text{if } r = \dfrac{2q^k-2}{q-1}, & k > 0 \\ 0 & \text{otherwise} & \end{array}$$

Proof. Same methods as the proof of lemma (2.4) will give the results. □

3. Some computations when $\deg \mathfrak{p} = 1$ or 2

First we write a table of the coefficients of s, s^2, s^{q+2} and s^{q^2+q+2} of $g^{q+1}\Delta$ and Δ^2.

(3.1)

	$g^{q+1}\Delta$	Δ^2
coefficients of s	-1	0
s^2	$[1]$	1
s^{q+2}	$[1]^2 - [1]^{q+1}$	$2[1]$
s^{q^2+q+2}	$[1]^3 - 4[1]^{q+1} + [1]^{2q+1}$	$-4[1]^{q+1} + 2[q]^2$

Let $\mathfrak{p} = (T+c)$ where $c \in \mathbf{F}_q$. Then

$$e_{\Lambda_{\mathfrak{p}}}(z) = z \prod_{\lambda \in \Lambda_{\mathfrak{p}}}{}' \left(1 - \frac{z}{\lambda}\right) = \frac{1}{T+c}\rho_{T+c}(z) = z + \frac{1}{T+c}z^q.$$

From Lemma, (2.2), (2.3) and (2.4), we get

(3.2)

	coefficient of X^{q-1}	coefficient of X^{2q-2} 2
G_{q-1}	1	0
$G_{2(q-1)}$	$\dfrac{-2}{T+c}$	1
$G_{(q+2)(q-1)}$	0	$\left(\dfrac{1}{T+c}\right)^q$
others	0	0

Using (1.4), (3.1) and (3.2) we get

$$\begin{aligned}
T_{(T+c)}(g^{q+1}\Delta) &= \{-(T+c)^{q-1} + 2[1](T+c)^{q-2}\}s + \{[a](T+c)^{2q-2} \\
&\qquad + [1]^2(T+c)^{q-2} - [1]^{q+1}(T+c)^{q-2}\}s^2 + \text{higher terms} \\
&= \{(T+c)^{q-1} + 2[1](T+c)^{q-2}\}g^{q+1}\Delta + \{[1](T+c)^{2q-2} \\
&\qquad + [1]^2(T+c)^{q-2} - [1]^{q+1}(T+c)^{q-1} - 2[1]^2(T+c)^{q-2}\}\Delta^2 \\
&= (T+c)^{q-2}[\{(T+c) + 2[1]\}g^{q+1}\Delta \\
&\qquad + \{[1]((T+c)^q - (T+c)) - [1]^2 - [1]^{q+1})\}\Delta] \\
&= (T+c)^{q-2}[\{(T+c) + 2[1]\}g^{q+1}\Delta + \{[1]^2 - [1]^2 - [1]^{q+1}\}\Delta^2 \\
&= (T+c)^{q-2}\{(2[1] + T + c)g^{q+1}\Delta - [1]^{q+1}\Delta^2\}
\end{aligned}$$

$$\begin{aligned}T_{(T+c)}(\Delta^2) &= -2(T+c)^{q-2}s + \{2(T+c)^{q-1}[1] + (T+c)^{2q-2}\}s^2\\ &= 2(T+c)^{q-2}g^{q+1}\Delta + (T+c)^{2q-2}\Delta^2\\ &= (T+c)^{q-2}\{2g^{q+1}\Delta + (T+c)^q\Delta^4\}.\end{aligned}$$

Hence the matrix form for T_{T+c} is

$$M_1 = (T+c)^{q-2}\begin{pmatrix} T+c+2[1] & 2 \\ -[1]^{q+1} & (T+c)^q \end{pmatrix} \tag{3.3}$$

Let $f = Ag^{q+1}\Delta + B\Delta^2$ be an eigenvector of M_1. If we compute $X = \frac{A}{B}$ by elementary linear algebra, then X is a root of the equation

$$[1]^{q+1}X^2 + (T^q - T)X + 2 = 0, \tag{3.4}$$

which is independent of c. And $\lambda' = \lambda/(T+c)^{q-2}$, where λ is an eigenvalue of M_1, satisfies

$$[\lambda' - \{(T+c) + 2[1]\}]\{\lambda' - (T+c)^q\} + 2[1]^{q+1} = 0.$$

But if we use the fact that $[1] = T^q - T$, λ satisfies
(3.5)

$$\lambda'^2 - (3T^q - T + 2C)\lambda' + (2T^{q^2+q} - 2T^{q^2+1} + T^{q+1} + 3cT^q - cT + c^2) = 0.$$

Applying the trivial case of Hensel's lemma ([5], p41), λ' is real, and so is λ.

Now let $\mathfrak{p} = (T^2 + aT + b)$, where $T^2 + aT + b$ is irreducible. Then

$$\begin{aligned}e_{\Lambda_\mathfrak{p}} &= z\prod_{\lambda\in\Lambda_\mathfrak{p}}\left(1 - \frac{1}{\lambda}\right) = \frac{1}{T^2+aT+b}\rho_{T^2+aT+b}(z)\\ &= z + \frac{T^q + T + a'}{T^2+aT+b}z^q + \frac{1}{T^2+aT+b}z^{q^2}.\end{aligned}$$

Again from lemmas (2.2), (2.3) and (2.4), we get

(3.6)

	coefficient of X^{q-1}	coefficient of X^{2q-2}
G_{q-1}	1	0
$G_{2(q-1)}$	$-2\cdot\dfrac{T^q+T+a}{T^2+aT+b}$	1
$G_{(q+2)(q-1)}$	$-2\cdot\dfrac{1}{T^2+aT+b}$	$\left(\dfrac{T^2+T+a}{T^2+aT+b}\right)^q$
$G_{(q^2+q+2)(q-1)}$	0	$\left(\dfrac{1}{T^2+aT+b}\right)^q$
others	0	0

As before, letting $p = T^2 + aT + b$,

$$\begin{aligned} T_{\mathfrak{p}}(g^{q+1}\Delta) &= p^{q-2}[\{p+2[1](T^q+T+a)+2([1]^2-(1)^{q+1})\}g^{q+1}\Delta \\ &\quad + \{[1]p^q + ([1]^2-[1]^{q+1})(T^q+T+a)^q + ([1]^{2q+1}-4[1]^{q+2} \\ &\quad + [1]^3) - p[1] - 2[1]^2(T^q+T+a) - 2([1]^3-[1]^{q+2})\}\Delta^2] \\ &= p^{q-2}[(-2T^{q^2+q}+2T^{q^2+1}+6T^{2q}-6T^{q+1}+2aT^q+T^2 \\ &\quad - aT+b)g^{q+1}\Delta + (-3T^q+T-a)[1]^{q+1} \end{aligned}$$

and

$$\begin{aligned} T_{\mathfrak{p}}(\Delta^2) =& p^{q-2}\{2(3T^q-T+a)g^{q+1}\Delta \\ &+ (-2T^{q^2+q}+2T^{q^2+1}+3T^{2q}-2T^{q-1}+aT^q+b)\Delta^2\} \end{aligned}$$

The matrix form for $T_{\mathfrak{p}}$ is

$$M_2 = p^{q-2}\begin{pmatrix} \begin{array}{c} -2T^{q^2+q}+2T^{q^2+1}6T^{2q} \\ -6T^{q+1}+2aT^3+T^2-aT+b, \end{array} & 2(3T^q-T+a) \\ -(3T^q-T+a)[1]^{q+1} & -2T^{q^2+q}+2T^{q^2+1}3T^{2q}-2T^{q+1}+aT^q+b \end{pmatrix}$$

If we compute the eigenvector $Ag^{q+1}\Delta + B\Delta^2$ of M_2 as before, we find that $X = \frac{A}{B}$ also satisfies

$$[1]^{q+1}X^2 + (T^q - T)X + 2 = 0$$

which is same as in the degree 1 case. Again by the trivial case of Hensel's lemma, the eigenvalues are real.

We now consider the case of weight $3q^2-3$ and $p = (T+c)$. The space of cusp forms of weight $3q^2-3$ is generated by $\{g^{2q+2}\Delta, g^{q+1}\Delta^2, \Delta^3\}$ over C. Then from the lemmas (2.2), (2.3), (2.4) and (2.5) we get

(3.7)

	coefficient of X^{q-1}	coefficient of X^{2q-2}	coefficient of X^{3q-3}
G_{q-1}	1	0	0
$G_{2(q-1)}$	$-2\frac{1}{T+c}$	1	0
$G_{3(q-1)}$	$3\cdot\left(\frac{1}{T+c}\right)^2$	$-3\cdot\frac{1}{T+c}$	1
$G_{(q+2)(q-1)}$	0	$\left(\frac{1}{T+c}\right)^q$	0
$G_{(q+3)(q-1)}$	0	$-3\left(\frac{1}{T+c}\right)^{q+1}$	$2\cdot\left(\frac{1}{T+c}\right)^q$
$G_{(2q+3)(q-1)}$	0	0	$\left(\frac{1}{T+c}\right)^{2q}$
others	0	0	0

Hence we need the coefficient of s, s^2, s^3, s^{q+2}, s^{q+3} and s^{2q+3} of $g^{2q+2}\Delta$, $g^{q+1}\Delta^2$ and Δ^3 to compute $T_\mathfrak{p}$ where $\mathfrak{p} = (T+c)$. The coefficients are given by

(3.8)

	$g^{2q+2}\Delta$	$g^{q+1}\Delta^2$	Δ^3
coefficients of s	-1	0	0
s^2	$2[1]$	1	0
s^3	$-[1]^2$	$-[1]$	-1
s^{q+2}	$3[1]^2 - 4[1]^{q+1}$	$4[1] - [1]^1$	3
s^{1+3}	$2[1]^{q+2} - [1]^3$	$[1]^{q+1} - 2[1]^2$	$-3[1]$
s^{2q+3}	$2[1]^{q+3} - [1]^{2q+2}$	$2[1]^{q+2} - [1]^3$	$-3[1]^2$

Then using (1.4), (3.7) and (3.8) we get

$$
\begin{aligned}
T_\mathfrak{p}(g^{2q+2}\Delta) &= (T+c)^{q-3}[\{(T+c)^2 + 4[1](T+c) + 3[1]^2\}g^{2q+2}\Delta \\
&\quad - \{4[1]^{q+1}(T+c) + 6[1]^{q+2}\}g^{q+1}\Delta^2 + [1]^{2q+2}\Delta^3] \\
T_\mathfrak{p}(g^{q+1}\Delta^2) &= (T+c)^{q-3}[\{2(T+c) + 3[1]\}g^{2q+2}\Delta + \{(T+c)^{q+1} \\
&\quad + 3[1](T+c)^q - [1]^q(T+c) - 3[1]^{q+1}\}g^{q+1}\Delta^2 \\
&\quad - [1]^{q+1}(T+c)^q\Delta^3] \\
T_\mathfrak{p}(\Delta^3) &= (T+c)^{q-3}[3g^{2q+2}\Delta + 6(T+c)^q g^{q+1}\Delta^2 + (T+c)^{2q}\Delta^3]
\end{aligned}
$$

Then the matrix form for $T_{(T+c)}$ with respect to the basis $\{g^{2q+2}\Delta, g^{q+1}\Delta^2, \Delta^3\}$ is

$$M = (T+c)^{q-3} \begin{bmatrix} (T+c)^2 + 4[1](T+c) + 3[1]^2, & 2(T+c) + 3[1], & 3 \\ -2[1]^{q+1}(2(T+c) + 3[1]), & (T+c)^{q+1} + 3[1](T+c)^q & 6(T+c)^q \\ & -[1]^q(T+c) - 3[1]^{q+1}, & \\ [1]^{2q+2}, & -[1]^{q+1}(T+c)^q, & (T+c)^{2q} \end{bmatrix}$$

Let $f = g^{2q+2}\Delta + Xg^{q+1}\Delta^2 + Y\Delta^3$ be an eigenvector of M_2. Then X (resp. Y) satisfies the equation

$$\{4(T+c)^{2q+1} + 6(T+c)^{2q}[1] + 4[1]^q(T+c)^2 + 12[1]^{q+1}(T+c) + 9[1]^{q+2} - 3[1]^q(T+c)^{2q}\} \cdot \{2X^3 + (2[1] + [1]^q)X^2 + 8[1]^{q+1}Y + 4[1]^{q+1}\} = 0$$

(resp.

$$\{4(T+c)^{2q+1} + 6[1](T+c)^{2q} + [1]^q(T+c)^2 + 6[1]^{q+1}(T+c)^q\} \cdot \{-6Y^3 + (2[1]^2 - 3[q]^{1+1})Y^2 + 2[q]^{2q+2}Y + [1]^{3q+3}\} = 0.)$$

Therefore one may conjecture that

CONJECTURE. For each k there exists a basis consisting of eigenforms in the space of Drinfel'd cusp forms of weight k and their eigenvalues are real algebraic.

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