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ON CONTINUED FRACTION EXPANSIONS OF $\sqrt{d}$

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The purpose of this paper is to study the continued fraction expansions of $\sqrt{d}$. Let d be a square-free positive integer congruent to 1 modulo 4. The only reason for this restriction is for the simplicity of our argument. Let N denote the norm from $\mathbf{Q}(\sqrt{d})$ to $\mathbf{Q}$. Let p_n/q_n denote the n-th convergent of $\sqrt{d}$. Let $\sqrt{d} = [a_0\,; a_1, a_2, \ldots]$ denote the continued fraction expansion of $\sqrt{d}$. We say that a positive integer k appears in $\sqrt{d}$ if $N(p_n - q_n\sqrt{d}) = k$ for some n and we say that a appears in the expansion of $\sqrt{d}$ if $a = a_n$ for some n.

We know that $\sqrt{d} = [a_0\,; \overline{a_1, a_2, \ldots, a_r = 2a_0}]$ [4],p.244. We get an easy upper bound for a_n by considering $N(a - b\sqrt{d})$. Suppose that $p/q = p_n/q_n$ is the n-th convergent of the expansion of $\sqrt{d}$. Then we have the inequality

$$\frac{1}{q_n(q_{n+1} + q_n)} < \left|\sqrt{d} - \frac{p}{q}\right| < \frac{1}{q_n q_{n+1}}$$

which implies

$$\frac{1}{(a_{n+1} + 2)q_n^2} < \left|\sqrt{d} - \frac{p}{q}\right| < \frac{1}{a_{n+1} q_n^2}.$$

So

$$\frac{1}{a_n + 2}\left|\sqrt{d} + \frac{p}{q}\right| < |q^2 d - p^2| < \frac{1}{a_n}\left|\sqrt{d} + \frac{p}{q}\right|.$$

But $|q^2 d - p^2|$ is at least one and $\sqrt{d}$ has periodic expansion. So for $n \gg 0$, we have $a_{n+1} < 2\sqrt{d}$. We just proved the following

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PROPOSITION 1. *Let* $\sqrt{d} = [a_0\,;\overline{a_1, a_2, \ldots, a_r = 2a_0}]$. *Then* $a_n < 2\sqrt{d}$ *for all* n.

Above proposition is well-known [6]. We know that $2a_0 = 2[\sqrt{d}]$ appears in the expansion of $\sqrt{d}$. Maybe $2a_0$ is the largest among a_n even if $2a_0 + 1 < 2\sqrt{d}$. Also it may appear only once in one period. But the author does not know the answer. On the other hand, suppose that k is square-free, $|k| < \sqrt{d}$, and $N(a - b\sqrt{d}) = k$ for some relatively prime integers a, b. Then

$$\left|\sqrt{d} - \frac{a}{b}\right| = \frac{k}{\left|\sqrt{d} + \frac{a}{b}\right|}\frac{1}{b^2}.$$

By a theorem of Lagrange [4],p.237, we know that a/b is a convergent of $\sqrt{d}$ if

$$2|k| < \left|\sqrt{d} + \frac{a}{b}\right|.$$

Let ε be the fundamental unit of the ring of algebraic integers of $\mathbf{Q}(\sqrt{d})$. Let $\varepsilon^n(a - b\sqrt{d}) = a' - b'\sqrt{d}$ for some integers a', b'. Then $N(a' - b'\sqrt{d}) = k$ for all even n. a', b' goes to infinity as n goes to infinity. So we see that k appears in $\sqrt{d}$.

PROPOSITION 2. *Suppose that* k *is square-free and* $|k| < \sqrt{d}$. *Also suppose that there are relatively prime integers* a, b *so that* $N(a - b\sqrt{d}) = k$. *Then* k *appears in* $\sqrt{d}$.

By proposition 2, we know that the period of $\sqrt{d}$ is at least

$$\sum_{1 \le k < \sqrt{d}} 1$$

where the summation is over all square-free k so that there are relatively prime integers a, b so that $N(a - b\sqrt{d}) = k$.

If we assume the generalized Riemann hypothesis, then we can prove that the class number h of the real quadratic number field $\mathbf{Q}(\sqrt{d})$ goes to infinity for certain types of d as d goes to infinity[1], [5]. Suppose that r is a fixed integer. Suppose that the period of $\sqrt{d}$ is not greater than r. This means that $r \geq \#\{p$ is a prime : $p < \sqrt{d}$, $N(a - b\sqrt{d}) = p$ for some relatively prime integers a, $b\}$. Let H denote the Hilbert class field of $\mathbf{Q}(\sqrt{d})$. Then by class field theory [3], we see

$$r \geq \frac{1}{2}\pi(\sqrt{d}, H/\mathbf{Q}(\sqrt{d}))$$

where $\pi(x, H/\mathbf{Q}(\sqrt{d}))$ is defined by

$$\pi(x, H/\mathbf{Q}(\sqrt{d})) = \#\{\mathfrak{p} : \text{principal prime ideal of } \mathbf{Q}(\sqrt{d}),\ N\mathfrak{p} \leq x\}.$$

Now by an effective version of Chebotarev theorem [2], there is a positive absolute constant c such that if the generalized Riemann hypothesis is true, then for every $x > 2$,

$$\left|\pi(x, H/\mathbf{Q}(\sqrt{d})) - \frac{1}{h}Li(x)\right| \leq c\left\{\frac{1}{h}\sqrt{x}\log(d^{2h}x^{2h}) + \log(d^{2h})\right\}$$

where

$$Li(x) = \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}$$

as $x \to \infty$. So

$$\left|\pi(x, H/\mathbf{Q}(\sqrt{d})) - \frac{1}{h}Li(x)\right| \leq c\{\sqrt{x}(2\log d + 2\log x) + 2h\log d\}$$

Suppose that there is a constant $\delta > 0$ so that $h < d^{1/4-\delta}$, for infinitely many d. Then for $x = \sqrt{d}$,

$$\begin{aligned}\left|\pi(\sqrt{d}, H/\mathbf{Q}(\sqrt{d})) - \frac{1}{h}Li(\sqrt{d})\right| &\leq c\{d^{1/4}(2\log d + 2\log\sqrt{d}) + 2h\log d\} \\ &\leq c(3d^{1/4} + 2h)\log d.\end{aligned}$$

As d goes to infinity, we get a contradiction if we assume that $\pi(\sqrt{d}, H/\mathbf{Q}(\sqrt{d}))$ is bounded. So we have

PROPOSITION 3. *Let r be a fixed integer. Suppose that*

$$\sqrt{d} = [a_0\,;\,\overline{a_1, a_2, \ldots, a_r = 2a_0}],$$

$d = 1 \bmod 4$, *and d is square-free. Then h goes to infinity as d goes to infinity.*

Siegel's lower bound of L-series gives that $h(d) \gg_r d^{1/2-\delta}$ for any $\delta > 0$ unconditionally [6]. Above result is weaker than this in that 1) it is conditional, 2) proposition 3 means $h(d) \gg d^{1/4-\delta}$ for any $\delta > 0$. But our approach is more transparent.

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