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WHAT ARE CHEVALLEY GROUPS FOR SL(2,F)–MODULE $V(m) \otimes_F V(n)$?

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0. Introduction

We presented an easy conjecture in [6]. This paper aims to exhibit its proof without difficulty. We shall follow standard conventions used in [3] through [6] unless otherwise stated.

Let F be any fixed algebraically closed field of characteristic zero. Let L be $sl(2,F)$ and $V = V(\lambda_1) \otimes V(\lambda_2)$ for $\lambda_1, \lambda_2 \in \Lambda^+$. Then it is not so hard to conjecture $G_V(K) \cong PSL_2(K)$ when λ_1 and λ_2 have the same parity, and $G_V(K) \cong SL_2(K)$ when otherwise.

It turns out that this is just an easy generalization of the previous result in [5].

1. Notations and Foundation

We assume that K is any extension field of the prime field F_p of characteristic $p > 0$ and that V is a faithful L-module for the time being ; then $\Pi(V)$, the weights of V, span a lattice $\Lambda(V)$ which lies between Λ_r and Λ.

For an admissible lattice M in V, its stabilizer L_v in L equals $H_v \oplus \amalg_{\alpha\in\Phi} Zx_\alpha$. So there arise natural monomorphisms $L_v(K) \hookrightarrow gl(V(K))$ and $L(Z) \hookrightarrow L_v$, the latter one also inducing a natural Lie algebra homomorphism $L(K) \longrightarrow L_v(K)$. A new $L(K)$-module $V(K)$ which

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may not be faithful is obtained through the homomorphism $L(K) \longrightarrow L_v(K)$.

2. Chevalley groups of type $\Lambda(V)$

We may define $L(K)$-automorphisms $\theta_\alpha(H)$ of $V(K)$ by the series $\sum_{t=0}^{\infty}(Hx_\alpha)^t/t!$ with the indeterminate H specialized to $c \in K$. The subgroup $G_v(K)$ generated by all $\theta_\alpha(c)$ is called a Chevalley group of type $\Lambda(V)$ in general ; adjoint if $\Lambda(V) = \Lambda_r$, and universal if $\Lambda(V) = \Lambda$. It is known that $L_v(K)$ and $G_v(K)$ are not dependent on V itself or on the choice of M, but dependent on the weight lattice $\Lambda(V)$.

3. Generalized results

Now we generalize the main theorem shown up in [5].

THEOREM. *Set $V = V(m) \otimes_F V(n)$ for $m, n \in \Lambda^+$ and $L = sl(2, F)$. Then we have $G_v(K) \cong PSL_2(K)$ when m and n have the same parity (even /odd), and $G_v(K) \cong SL_2(K)$, when otherwise.*

Proof: Letting $\{v_i : i = 0, 1, \cdots, m\}$ and $\{\bar{v}_j : j = 0, 1, \cdots, n\}$ be chosen bases as in [6], we know that for $m \leq n$

$$V(m) \otimes V(n) = V(m - 2m + n) \oplus V(m - 2m + n + 2) \oplus \cdots \oplus V(m + n)$$

As we are well aware,

$\Lambda(V) = \Lambda_r$ if and only if m and n have the same parity ;

$\Lambda(V) = \Lambda$ if and only if m and n do not have the same parity.

Note also $V(\lambda)$ has a minimal admissible lattice $M^\lambda_{\min}$ with a Z-basis $B^\lambda = \{v^\lambda_0, \cdots, v^\lambda_\lambda\}$ for which the following formulas hold :

$$h \cdot v^\lambda_i = (\lambda - 2i)v^\lambda_i, \; xv^\lambda_i = (\lambda - i + 1)v^\lambda_{i-1} \text{with } v^\lambda_{-1} = 0,$$
$$y \cdot v^\lambda_i = (i + 1)v^\lambda_{i+1} \text{with } v^\lambda_{\lambda+1} = 0.$$

Since $V = V(m) \otimes V(n) = V(m-2m+n) \oplus V(m-2m+n+2) \oplus \cdots \oplus (m+n)$, we may decompose $V(K)$ as follows:

$$\begin{aligned} V(K) = M_{\min} \otimes_Z K = \{&(Zv_0^{-m+n} \oplus Zv_1^{-m+n} \oplus \cdots \oplus Zv_{-m+n}^{-m+n}) \oplus \\ &\cdots \oplus (Zv_0^{m+n} \oplus \cdots \oplus Zv_{m+n}^{m+n})\} \otimes_Z K \\ = (&M_{\min}^{-m+n} \oplus M_{\min}^{-m+n+2} \oplus \cdots \oplus M_{\min}^{m+n}) \otimes_Z K, \end{aligned}$$

whence we have correspondences as

$$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \longrightarrow M_B^B(\theta_2(c))$$
$$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \longrightarrow M_B^B(\theta_{-2}(c)),$$

where $B = \dot{U}B^\lambda$. By making use of [5], we easily see that $G_v(K) \cong SL_2(K)$ when $\Lambda(V) = \Lambda$, i.e., m and n don't have the same parity, and $G_v(K) \cong PSL_2(K)$, when $\Lambda(V) = \Lambda_r$, i.e., m and n do have the same parity.

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