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INSTANTON INVARIANTS ON 4–MANIFOLDS

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0. Introduction

In [2] Donaldson introduced polynomial invariants for smooth, closed and simply connected 4-manifolds. The polynomial invariants are defined by evaluating certain rational cohomology class with the moduli space of equivalence classes of instantons in the Banach manifold B of equivalence classes of connections. Also in [3] he introduce 2-torsion polynomial invariant for closed simply connected spin 4-manifolds. In section 1 we summerize the rational cohomology groups of the orbit space of B. In section 2 we introduce the compactification of the moduli space, definition of polynomial invariant and Donaldson's main theorems. In section 3 as examples of indecomposiabity we considered complex algebraic surfaces. In section 4 we introduce the definition of 2-torsion instanton invariants for spin 4-manifolds and the stable range condition for compactness. We introduce the Fintushel and Stern Theorem, in [4] they gave a relation between rational polynomial invariants and 2-torsion polynomial invariants by connected sum with $S^2 \times S^2$. Finally we investigate the polynomial invariants on the space of connected sum. Theorem 11 is a special case of a Donaldson Theorem, however it is useful to our Thorem 12; the space $I(\lambda)$ obtained from the moduli space cutting out by codimension 2 submanifolds is the compact 4-manifold consisting of finite copies of $S^1 \times SO(3)$.

1. Rational Cohomology of the orbit space $\tilde{B}$ and B^*

Let P be an $SU(2)$ principal bundle over a closed oriented simply connected 4-manifold M. The principal bundle is determined by the

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Chern number $k =< c_2(P), [M] >$ up to isomorphism. The Yang-Mills energy of an instanton over M is a topological characteristic number of the bundle P carrying the connection. If P admits any anti-self-dual connection then k must be non-negative. For each $k \geq 0$ we have a moduli space M_k of anti-self-dual connections on P modulo gauge equivalence, and M_0 consists of a single point representing the product connection on the trivial bundle since M is simply connected.

Let A be an irreducible anti-self-dual connection. The self-dual part of the curvature, $F^+(A) = 0$ where $F^+ = \frac{1}{2}(F + *F)$. The curvature of another connection $A+a$ can be written $F(A+a) = F(A)+d_Aa+\frac{1}{2}[a,a]$. Taking the self-dual part we have $F^+(A+a) = {d_A}^+a + \frac{1}{2}[a,a]^+$. The moduli space M_k is obtained by dividing the solutions of this equation by the action of the gauge transformation group $\mathcal{T} = \operatorname{Aut} P$. For small deformations can be replaced by imposing the Coulomb gauge condition ${d_A}^+a = 0$, which defines a local transversal slice for the action of $\mathcal{T}$. A neighbourhood of the point $[A]$ in the moduli space M_k is given by the sloutions of the differential equations

$$\begin{cases} {d_A}^+a = 0 \\ {d_A}^+a + \frac{1}{2}[a,a]^+ = 0. \end{cases}$$

These are non-linear first order equations, the non-linearlity coming from the quadratic term $[a,a]^+$. The linearization at A can be written as an operator ${d_A}^* \oplus {d_A}^+ : \ \Omega^1(adP) \to \Omega^0(adP) \oplus \Omega^2 + (adP)$ which is elliptic. The index of this operator ${d_A}^* \oplus {d_A}^+$ is given by the formula $m = 8k - 3(1 + b^+(M))$, where $b^+(M)$ is the dimension of a maximal positive subspace of the intersection form on $H^2(M)$. The number m is the virtual dimension of the moduli space M, for a generic Riemannian metric on M the part of the moduli space consisting of irreducible connections will be a smooth manifold of dimension m.

Assume that $b^+(M) > 0$. Then it can shown that for generic metrics and $k \geq 1$ every instanton is irreducible. A reducible anti-self-dual connection on P corresponds to an element $c \in H^2(M;R)$ which is in the intersection of the integer lattice and the subspace $H^- \subset H^2(M:R)$ consisting of calsses of anti-self-dual forms. The codimension of H^- is

b^+, so if $b^+ > 0$ and H^- is in general position. There are no non-zero classes in the intersection. On the same lines we can show that if $b^+ > 1$, then for generic 1-parameter families of Riemannian metrics on M we do not encounter any nontrivial reducible connections.

Let B^* be the space of all irreducible connections of P modulo gruge equivalence. It is an infinite dimensioanl manifold and under our assumptions the moduli space M_k is a submanifold of B^*, for generic metrics on M. The differential topological invariants of the 4-manifold M are defined by the pairings of the fundamental homology class of the moduli space M_k with the cohomology classes of B^*. The moduli space M_k certainly depends on the choice of metric, so let us write $m_k(g)$ for the moduli space defined with respect to a metric g on M. Suppose g_0, g_1 are two generic metrics on M. We join them by a smooth path g_t, $t \in [0,1]$ of metrics. If $b^+ > 1$, then we do not encounter any reducible connections so we can define $w = \{([A],t) \in B^* \times [0,1] | [A] \in M_k(g_t)\}$. For a gemeric path g_t the space W is a manifold-with-boundary the boundary consisting of the disjoint union of $M_k(g_0)$ and $M_k(g_1)$. Fix a bases point in M and let $\tilde{B}$ be the $SO(3)$ bundle over B^* whose points represent equivalence classes of connections on a bundle witch is trivialized over the base point. The space $\tilde{B}$ is weakly homotopy equivalent to the space $\mathrm{Map}(M, BG)$ of based maps of degree k from M to the classifying space BG of the structure group $G = SU(2)$. One can show that the rational cohomology of $\tilde{B}$ is a polynomial algebra on 2-dimensional cohomoloyg classes corresponding by the 2-dimensional homology of M. That is, the cohomology is generated by the image of a natural map $\tilde{\mu} : H_2(M : Z) \to H^2(\tilde{B} : Z)$ which is just the slant product in $\mathrm{Map}\, s(M, BG) \times M$ with the 4-dimensional class pulled back from the generator of $H^4(BG)$ under the evaluation pairing $\mathrm{Maps}(M, BG) \times M \to BG$. This map $\tilde{\mu}$ descends to map $\mu : H_2(M : Z) \to H^2(B^* : Z)$. The fibration $SO(3) \to \tilde{B} \to B^*$ give the Gysin sequence :

$$\to H^{n-4}(B^* : Q) \to H^n(B^* : Q) \to H^n(\tilde{B} : Q) \to H^{n+1-4}(B^* : Q) \to \cdots$$

THEOREM 1. *(a)* $H^*(\tilde{B} : Q) = Q[\tilde{\mu}(\alpha_1), \cdots, \tilde{\mu}(\alpha_\lambda)]$
(b) $H^*(B^* : Q) = Q[e, \mu(\alpha_1, \cdots, \mu(\alpha_\lambda)]$
where $\{\alpha_1, \cdots, \alpha_\lambda\}$ *is a basis of* $H^2(M : Z)$, $e = \mu(a)$, $< a >= H_0(M)$.

2. Rational Instanton Invariants

In general the moduli spaces are not compact, we should compactify them to get the fundamental homology class. The compactification $\bar{M}_k$ of M_k is a subset of $M_k \cup M_{k-1} \times M \cup M_{k-1} \times s^2(M) \cup \cdots \cup s^k(M)$. The topology is defined by a notion of convergence. If $(x_1, \cdots, x_l)$ is a point in the symmetric product $s^l(M)$, a sequence converges (up to equivalence) away from $x_1, \cdots x_l$, and the energy desity $|F(A_n)|^2$ converge as measures to $|F(A)|^2 + 8\pi^2 \sum_{i=1}^{l} \delta x_i$. The closure $\bar{M}_k$ of $M_k$in this topolohgy is compact.

If the moduli space M_k has even dimension $m = 2d$, then for each k such that $4k > (2b^+(M) + 3)$ there is a natural pairing between the moduli space M_k and a product of cohomology $\mu(\alpha_1) \cup \cdots \cup \mu(\alpha_d)$ for any $\alpha_1, \cdots, \alpha_d \in H_2(M)$. To define the pairings we should extend $\mu(\alpha)$ to $\bar{\mu}(\alpha) \in H^2(\bar{M}_k)$. For $l > 0$ and $\alpha \in H^2(M)$ let $s^l(\alpha) \in H^2(s^l(M))$ be the natural symmetric sum of copies of α and let $a^{(l)} = {\pi_1}^*\mu(\alpha) + {\pi_2}^* s^l(\delta) \in H^2(M_{k-l} \times s^l(M))$, where δ is the Poincare dual of α. Then the extension $\bar{\mu}(\alpha)$ of $\mu(\alpha)$ tp $H^2(\bar{M}_k)$ is $a^{(l)}$ on $\bar{M}_k \cap (m_{k-l} \times s^l(M))$. For any $\alpha_1, \cdots, \alpha_d \in H^2(M : Z)$, $\bar{\mu}(\alpha_1) \cup \cdots \cup \bar{\mu}(\alpha_d) \in H^{2d}(\bar{M}_k)$ and we can define a pairing $< \bar{\mu}(\alpha_1) \cup \cdots \cap \bar{\mu}(\alpha_d), [\bar{M}_k] >$. Note that if the strata $\bar{M}_k \cap (M_{k-l} \times s^l(M))$ making up $\bar{M}_k$ have codimension 2 or more, the $[\bar{M}_k]$ is the fundamental homology class for $l > 0$.

$$\begin{aligned} &\dim(M_{k-1} \times s^l(M)) = \dim M_{k-l} + 4l = \dim M_k - 4l \text{ if } l < k \\ &\dim s^k(M) = 4k \qquad\qquad\qquad\qquad\qquad\qquad \text{if } l = k. \end{aligned}$$

Since b^+ is odd the condition for $S^k(M) = 4k$ to have codimension 2 is that $8k - 3(1 + b^+(M) > 4k$, which is the stable range condition. The same pairing can be defined by the other procedure. For a generic surface Σ in M the retriction of any irreducible anti-self-dual connection over M to $\sum$ is again irreducible, we have restriction map $r : M_j \to B^{\Sigma}*$. If α is the fundamental class of Σ in $H_2(M)$ the cohomology class $\mu(\alpha)$ is pull back from B_{Σ^*} by r^*. We choose a generic codimension 1 submanifold in the target space which represents by the cohomology class and let V_Σ be the preimage of this in the moduli space. Let $\Sigma_1, \cdots, \Sigma_d$ be surfaces in M, in general position the intersection $M_k \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_j}$ is compact if $4k > 3(1 + b^+(M))$. Note that we can choose the V_{Σ_i} so that all the intersections in all the moduli spaces are transverse. If $\{A_n\}$ is a sequence in $V_{\Sigma_i} \subset M_k$ which converges to $([A],\ x_1 \cdots x_l)$ and if none of the points x_i lies in Σ_j then the limit $[A]$ is in $V_{\Sigma j} \subset M_j$. This intersection number is independent of the choice of Riemannian metric on M, and of the choice of V_{Σ_i} and of the sufaces Σ_1 within their homology classes α_i.

THEOREM 2[1]. *Let M be a closed, compact and simply connected 4-manifold with $b^+(M) > 1$ odd and $4k > (3b^+(M) + 3)$. The map $q_{K,M}$: $s^d(H^2(M : Z)) \to Z$ given by $q_{K,M}(\Sigma_1 \cdots \Sigma_d) = \#(M_k \cap V_1 \cap \cdots \cap V_d)$, counted with sign where $d = 4k - \frac{3}{2}(1 + b^+(M))$, is a diffeomorphic invariant up to sign, natural with respect to orientation preserving diffeomorphisms.*

REMARK. Let B_k^* be the space of irreducible connections modulo equivalence on a bundle of chern class k. Define a topology on the union

$$\bar{B}_k^* = B_k^* \cup B_{k-1}^* \times M \cup B_{k-2}^* \times s^2(M) \cup \cdots$$

defining that a wequence $\{A_n\}$ converges to $([A], x_1 \cdots x_l)$ if
(a) the connections converge away from the x_i.
(b) the self-dual parts $|F^+(A_n)|^2$ of the energy densities are uniformly bounded.
(c) the Chern-Weil integrands $\mathrm{Tr}(F(A_n)^2)$ converges as measures to the limit $\mathrm{Tr}(F(A)^2) + 8\pi^2 \sum_{i=1}^{l} \delta_{x_i}$.

THEOREM 3 [2]. *Let M be a 4-manifold which satisfies the condition of Theorem 2. If M can be written as a smooth, oriented, connected sum $M = M_1 \# M_2$ and each of the numbers $b^+(M_1) > 0$, then $q_{K,M}$ is identically zero for all k.*

THEOREM 4 [2]. *Let M be a complex algebraic surface and let $\alpha \in H_2(M)$ be the Poincare dual to the Kahler class $[w]$ over the surface M. Then for all large k the invariant $q_{K,M}(\alpha^d) > 0$.*

3. Complex Algebraic Surfaces

Let V_n be a non-singular hypersurface of $\mathbf{C}P^3$ with degree n. By Lefschetz's Theorem V_n is simply connected. For instance, $V_1 = \mathbf{C}P^2$, $V_2 = S^2 \times S^2$, $V_3 = \mathbf{C}P\#6\mathbf{C}P$, $V_4 = K_3$-surface, and the followings are homotopically equivalent,

$$V_{2m} \cong r_m\mathbf{C}P^2\#t_m\overline{\mathbf{C}P^2}, \text{ where } \begin{cases} r_m = \frac{2}{3}[(2m+1)(2m^2-4m+3)] - 1 \\ t_m = \frac{2}{3}[m(8m^2+1)] \end{cases}$$

$$V_2m \cong a_mV_4b_mV_2, \text{ where } \begin{cases} a_m = \frac{1}{6}[m(m^2-1)] \\ b_m = \frac{1}{6}[(m-2)(13m^2-22m+3)] - 1 \end{cases}$$

By Freedman's classification theorem for the simply connected compact topological 4-manifolds we have the followings.

THEOREM 5. *(a) V_{2m+1} is homeomorphic to $r_m\mathbf{C}P^2\#t_m\overline{\mathbf{C}P}^2$.*
(b) V_{2m} is homeomorphic to $a_mV_4\#b_mV_2$.
(c) By Theorem 3.4, in (a), (b) we cannot replace the homeomorphisms by the diffeomorphisms, where V_n for $n \geq 5$.

A K_3-surface V_4 is a compact, simply connected complex surface with trivial canonical bundle. All K_3-surfaces are diffeomorphic but not necessary biholomorphic. Some K_3-surfaces are elliptic surfaces. There is a holomorphic map $\pi : V_4 \to \mathbf{C}P^1$ whose generic fibre is an elliptic cureve $T^2 = S^1 \times S^1$. From V_4 we can construct a family of complex surfaces $S_{p,q}$, $p, q > 1$ by performing logarithmic transformations to a pair of generic fibers of π with multiplicities p and q. From a differential topological point of view a logarithmic transform of multipicity p is performed as follows; Let D^2 be a small disc and $n^{-1}(D^2) = S^1 \times S^1 \times D^2$ and $c = \partial D^2 = S^1$ on $\pi^{-1}(D^2)$. Then $\partial\pi(D^2) = \partial(V_4 \setminus \pi^{-1}(D^2) = S^1 \times S^1 \times S^1 = T^3\#$. Let A, B_1, B_2 be the simply closed curves generating $H_1(T^3)$ such that $A = \partial D^2$. Let h be a diffeomorphism of $\partial\pi^{-1}D^2 \to \partial(V_4 \setminus \pi^{-1}D^2)$ which takes $h(c) = pA + \lambda_1B_1 + \lambda_2B_2$. We call $S_p \equiv (V_4 \setminus \pi^{-1}(D^2))\bigcup_h \pi^{-1}(D^2)$ the logarithmic transformation of V_4. The $S_{p,q}$ are again elliptic surfaces and are diffeomorphism types realised within the one homotopy class as V_4 from investigating the $S_{p,q}$.

4. 2-Torsion Instantion Invariants

We introduce mod 2 cohomology classes using indices of operators which make essential use of a spin structure on the manifold. If M is a spin 4-manifold there are spin bundles S^+, S^- corresponding to the fundamental representation of the two factor $\mathrm{Spin}(4) = \mathrm{SU}(2) \times SU(2)$ consider the Dirac operator $\mathcal{D}_A : \Gamma(S^-) \to \Gamma(S^+)$. For each connection A we can associate an extended Dirac operator $\mathcal{D}_A : \Gamma(S^- \otimes_{\mathbf{C}} E) \to \Gamma(S^+ \otimes_{\mathbf{C}} E)$.

Since $\mathrm{SU}(2) \cong \mathrm{SP}(1)$, each of structures and compatible with the Dirac operator and so the kernel and cokernel of the operator $\mathcal{D}_A$ are naturally real vector spaces. Thus the index of the family of these operators gives a real virtual bundle $\mathrm{Ind}\,\mathcal{D}_A \in \mathrm{KO}(B)$. For spin 4-manifold M we define cohomology classes $u_i = w_i(\mathrm{Ind}\,\mathcal{D}_A) \in H^1(B : Z/2)$. The numerical index of the coupled operator compares with that of the Dirac operator $\mathbf{C}$ by

$$\mathrm{Ind}\,\mathcal{D}_A = c_2(E) + 2.$$

It follows that $(-1) \in \mathcal{T}$ acts trivially on $\mathrm{Ind}\,\mathcal{D}_A$ when $c_2(E)$ is even. In this case the bundle descends to a line bundle $\mathrm{Ind}\,\mathcal{D}_A \to B^*$.

The next theorem is well known.

THEOREM 6. *Let M be a closed simply connected 4-manifold.*
(1) If M is spin and $c_2(E)$ is even, then $\pi_1(B^*) = Z_2$.
(2) Neither M is spin nor $c_2(E)$ is even, then $\pi_1(B^*) = 0$.

Now consider a simply connected spin 4-manifold M with $b^{+(M)}$ even. The moduli space of anti-self-dual connections of the SU(2)-bundle over M with $c_2(E) = k$ has its virtual dimnsion $8k - 3(1 + b^+(M)) = 2d + r$. Let homology classes $z_1 \cdots z_d \in H_2(M : Z)$ be represented by generic surfaces $\Sigma_1 \cdots \Sigma_d$.

THEOREM 7. *If $4k > 3(1 + b^+(M)) + r$, then the intersection $V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap M_k$ is a compact r-manifold in B_k^*, for a generic metric on M, where $r = 0, 1, 2, 3$.*

Proof. Let $I_r = V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \cap M_k$. Suppose that $\{A_\alpha\}$ is a sequence in I_r. There is a subsequence of $\{A_\alpha\}$ which cinverges to

$([A],(x_1,\cdots,x_l))$ in $\bar{M}_k$. There are at most $2l$ of the surfaces V which contain one of the points x_i, so $[A]$ must lie in at least $d-2l$ of the V_{Σ_i}. For $0<l<k$ $\dim M_{k-l}=2d+r-8l\geq 2(d-2l)$ since $[A]$ lies in $d-2l$ of the V_{Σ_i}. Hence $r\geq 4l$. If $r\leq 3$, then $l=0$. So $[A]$ is a limit point of the sequence in I_r. If $l=k$, so A is flat, $[A]$ does not lie in any of the V_{Σ_i}, so we have $d\leq 2k$, that is, $4k\leq 3(1+b^+(M))+r$. Since $4k\leq 3(1+b^+(M))+r$. This case does not occur.

THEOREM 8. *For any two generic metrics on M, the intersections are cobordant in B^* if $r\leq 2$.*

Proof. Suppose g_0, g_1 are two generic metrics on M. Join them by a smooth path g_t, $t\in[0,1]$ of metrics. If $r\leq 3$, $I_r=V_{\Sigma_1}\cap\cdots\cap V_{\Sigma_d}\cap M_k$ is compact for each G_t.

Let $N=\{([A],t)\in B^*\times[0,1]|[A]\in m_k(G_t)\}$, and

$$V_i=\{([A],t)\in B^*\times[0,1]|[A]\in \underset{\Sigma_i}{V}(g_t)\}$$

$I=N\cap V_1\cap\cdots\cap V_d$ has $(r+1)$-dimension. If $r+1\leq 3$ then I is compact, i.e. $r\leq 2$ and with boundary $\partial I=I_r(G_0)\cup -I_r(g_1)$. For a generic path g_t the space I is a $r+1$ manifold-with-boundary consisting of the disjoint union of $I_r(g)$ and $I_r(G_1)$.

Note that the group of orientation preserving self-homotopy equivalences of M acts natrually on the cohomology of B^*. If a class $\sigma\in H^2(B^*)$ is fixed by this action, the we call such a class σ an invariant class.

THEOREM 9 [3]. *Let M be a compact, smooth, oriented, and simply connected 4-manifold with $b^+(M)>1$. Let σ be an invariant class in $H^r(B^*,R)$ for $r\leq 2$. If $rk>3(1+b^+(M)+r$ and the dimension $M_k=8k-3(1+b^+(M)=2d+r$, then the map $q_{K,\sigma,M}:H_2(M,Z)\times\cdots\times H_2(M,Z)\to R$ given by $q_{K,\sigma,M}(\Sigma_1\cdots\Sigma_d)=<\sigma,\ M_k\cap V_{\Sigma_1}\cap\cdots\cap V_{\Sigma_d}>$ is up to sign a diffeomorphic invariant of M, and natural with respect to orientateion preserving diffeomorphisms.*

REMARK. (a) Let M be a simpley connected spin 4-manifold with $b_M{}^+$ even. The $\dim M_k=8k-3(1+b^+)=2d+1$ and $4k>3(1+b^+)+1$. Then $V_{\Sigma_1}\cap\cdots\cap V_{\Sigma_d}\cap M_k$ is a compact 1-manifold $w_1=w_1(\det\ \mathrm{Ind}\,\mathcal{D}_A)\in$

$H^1(B_k{}^* : Z_2)$ if k is even, then $q_{k,w,M}(\Sigma_1 \cdots \Sigma_d) =< w_1, M_k \cap V_{\Sigma_1} \cap V_{\Sigma_d} >\in Z_2$.
(b) Simillary let M be spin and b^+ : odd and k : odd, we have $w_2 \in H^2(B^*, Z_2)$ as in [1]. The $\dim M_k = 8k - 3(1 + b^+) = 2d + 2$. Then the map $q_{k,w_2,M}$: $\mathrm{Sym}^d H_2(M : Z) \to Z_2$ given by

$$q_{k,w_2,M}(\Sigma_1 \cdots \Sigma_d) =< \mu(\Sigma_1) \cup \cdots \mu(\Sigma_d) \cup w_2, M_k > .$$

THEOREM 10 [4]. *Let M be a closed simply connected spin 4-manifold with a Donaldson polynomial $q_{l,M}$ fo degreed where l is odd. Then $q_{l+1,w_1,M\#S^2\times S^2}$ is defined and for any $\Sigma_1 \cdots \Sigma_d \in H_2(M : Z)$ and for $x = S^2 \times 0$, $y = 0 \times S^2$ in $H_2(S^2 \times S^2; Z)$ we have $q_{l,M}(\Sigma_1 \cdots \Sigma_d) = q_{l+1,w_1,M\#S^2\times S^2}(\Sigma_1, \cdots, \Sigma_d, x, y)$ (mod 2)*

5. Connected Sums

Let $X = X_1 \# X_2$ be a smooth connected sum. Fix generic g^i on the space X_i with injective radius of $X_i > 1$. Choose points x_i in X_i end $e_i : T_{x_i}X_i \to X_i$ is the exponential map. Let ϕ be an orientation reversing isometry, $\phi : T_{x_1}X_1 \to T_{x_2}X_2$. Form the connected sum $X = X_1 \# X_2$ by identifying $e_1(\xi) \in X_1$, $0 < \lambda < |\xi| < 1$ with $e_2((\lambda/|\xi|)\phi(\xi)) \in X_2$ and cutting out the λ-balls about x_i. Let τ_λ be a cut off function on X_1.

$$\tau_\lambda(x) = \begin{cases} 0, & d(x_1, x) \le \lambda \\ \frac{1}{2}, & d(x_1, x) = \lambda \\ 1, & d(x_1, x) \le \lambda \end{cases}$$

Define a metric g_λ on X by

$$g_\lambda = \begin{cases} \tau_\lambda g^1 + (1 - \tau_\lambda)(e_2 \phi e_1{}^{-1})^* g^2 \text{ on } X_1 \backslash B(x_1, \lambda) \\ g^2 \text{ on } X_2 \backslash B(x_2, 1) \end{cases}$$

X contains isometric compies of $X_i \backslash B(x_i, \lambda^{\frac{1}{4}})$.

The riemannian manifold (X, g_λ) has a neck of radius λ. As $\lambda \to 0$ we have $(X = X_1 \# X_2, g_\lambda) \to (X_1 \vee X_2,\ g^1 \vee g^2)$.

THEOREM 11. *Suppose M be a closed, smooth, oriented, and simply connected 4-manifold and let $X = M\#n\mathbf{C}P^2$ be a smooth connected sum, $4k > 3(1+b^+(M))$.*

Then $q_{K,M} = q_{K,M\#n\overline{\mathbf{C}P}^2}$ on $S^d(H_2(M))$ where $8k-3(1+b_2{}^+(M)) = 2d$.

Proof. Choose d subspaces $\Sigma_1 \cdots \Sigma_d$, in general position, in $M\backslash$ball where $2d = \dim M_{K,X} = \dim M_{K,M}$. Fix representatives $V_1 \cdots V_d$ such that all multiple intersections with the (M,g) moduli spaces are transverse. If λ is small, the intersections:

$$I(\lambda) = M_{K,X}(g) \cap V_1 \cap \cdots \cap V_d$$

$I = M_{K,M}(g) \cap V_1 \cap \cdots \cap V_d$ can be identified as sets. In fact for given a point in I we can construct a point in $I(\lambda)$ using $A = \theta$(the product connection on $n\overline{\mathbf{C}P}^2$). The transverse intersection of the V_i with $M_{K,M}(g)$ goes over to a single transverse intersection with $M_{K,X}(g_\lambda)$. The points are counted with the same sign by definition of the orientation. Conversely we need to show that for a sufficiently small λ every point of $I(\lambda)$ can be represented by a point of I and flat connection θ on $n\overline{\mathbf{C}P}^2$. Suppose that $\lambda_\alpha \to 0$ and $\{A_\alpha\}$ is a sequence in $I(\lambda_\alpha)$. We may suppose the sequence A converges to A^1 and A^2 on $n\overline{\mathbf{C}P}^2\backslash$pts, $M\backslash$pts with exceptional sets of sizes l_1, l_2, $l_2+k_2 \le l_2+k_2+l_1+K_1 \le k$. Since $4k > 3(1+b^+)$, we have $l_2 = 0$ and $k_2 = k$. hence $1_1 = 0$ and $A^1 = \theta$.

Suppose $X = M\#n\overline{\mathbf{C}P}^2$ is a smooth, oriented, connected sum with $b^+(M) > 1$ odd and $4k > 3(1+b^+(x))+4$. Let $8k-3(1+b^+(x)) = 2d+4$. We fix a partition $d = d_1+d_2$ and homology class $[\Sigma_1]\cdots[\Sigma_d] \in H_2$ (M,Z) which are represented by surfaces Σ_i in M and $[\Sigma_1{}'],\cdots,[\Sigma_d{}'] \in H_2(n\overline{\mathbf{C}P}^2 : Z)$. Assume that $2d_1 > 3(1+b^+(M))$, $2d_2 > 3$. We define k_1, k_2 by $8k_1 - 3(1+b^+(M)) = 2d_1$, $8k_2 - 3 = 2d_2+1$. Let $V_1,\cdots,V_{d_1}$ and $V_1{}',\cdots,V_{d_2}{}'$ be codimension 2 submanifolds corresponding to the surfaces $\Sigma_1\cdots\Sigma_{d_1}$ and $\Sigma_1{}'\cdots\Sigma_{d_2}{}'$ respectively. Consider a family of metrics $g(\lambda)$ on X as before, with the neck diameter $o(\lambda^{\frac{1}{2}})$ and converging to given generic metrics $g^1 g^2$ on M and $n\overline{\mathbf{C}P}^2$ respectively. Let $I(\lambda) = M_{K,X}(g) \cap V_1 \cap \cdots \cap V_d \cap V_1{}' \cap \cdots \cap V_d{}'$. Under these assumptions we have the following theorem.

THEOREM 12. *$I(\lambda)$ is a disjoint union of finite copies of $S \times SO(3)$ for small λ.*

Proof. Let $I_1 = M_{k_1,M} \cap V_1 \cap \cdots \cap V_d$. Since $2d_1 > 3(1+b^+(M))$ I_1 is a finite set of q irreducible self-dual connections for generic metric g^1 on M. Let $I_2 = M_{K_2} \cap V_1' \cap \cdots \cap V)d'$. Since $2d_2 > 3$ and $8k_2 = 2d_2 + 4$ I_2 is a compact 1-dimensional manifold for a generic metric g^2 on $n\overline{\mathbf{C}P}^2$. Thus I_2 is a disjoint union of circles because we are in the stable range. Let $A_1 \in I_1$ and $A_2 \in I_2$. The gluing procedure shows that for small λ, there is a family of anti-self-dual connections over X parametrized by a copy of SO(3), namely the gluing parameter, and neighbourhoods of the points A_i in their respective moduli spaces. Taking the intersection with the V_i and V_j' is the same as removing these two latter sets of parameters in the family. We obtain a copy (A_1, A_2) of SO(3) in the intersection $I(\lambda)$. A point in I_1 and a component of I_2 form a complete connected component $S^1 \times$ SO(3) of $I(\lambda)$. The sets l_1, l_2 have finite components of points and circles respectively. $I(\lambda)$ contains the disjoint union of $|I_1| \cdot$ (# of components of I_2) copies of $S' \times$ SO(3). On the other hand, suppose that we have a sequence $\lambda_n \to 0$ and a sequence $\{A_n\}$ of connections in $I(\lambda_n)$. By Ublenbeck weak compactness theorem, after taking a subsequence we can suppose that the subsequence $\{A_n\}$ converges to limits B_1, B_2 over the complement of exceptional sets of sizes l_1, l_2 in the two manifolds M and $n\overline{\mathbf{C}P}^2$. Where B_1 and B_2 are anti-self-dual connections on bundles with chern numbers k_1, k_2 over M and $n\overline{\mathbf{C}P}^2$ respectively. Then we have an energy inequality $k_2 + k_2 + l_1 + l_2 \leq k$. By the stable range conditions at least one of the k_i must be strictly positive. Suppose that k_2 is zero. B_2 is the trivial flat connection. Then each surface Σ_j' must contain one of the l_2 exceptional points in $n\overline{\mathbf{C}P}^2$. So $d_2 \leq 2l_2$.

Over the $M_{k_1,M}$, at least $d_1 - 2l_1$ of the V_i must meet the modull space $M_{K_1,M}$, so $s(d_1 - 2l_1) \leq 8k_1 - 3(1 + b^+(X)) - 8l_1$. Since $2d_1 > 3(1 + b^+(M))$ we have contradiction. Thus we have $k_2 \neq 0$ and $l_1 = 0$. Similaries we have $k_1 \neq 0$ and $l_2 = 0$. Thus we have $k = k_1 + k_2$, $l_1 + l_2 = 0$ and so, $B_1 \in I_1 = M_{k_1,M} \cap V_1 \cap \cdots \cap V_d$ and $B_2 \in I_2 = M_{k_2, n\overline{\mathbf{C}P}^2} \cap V_1' \cap \cdots \cap V_d'$ and B_2 is contained one of the circles. It follows that for large n the point $A(n)$ lies in the component $S^1 \times$ SO(3), for small λ.

REMARK. In Theorem 7, under the stable rang condition $4k > 3(1+b^+)+r$ if $r \leq 3$, then $I(\lambda)$ is compact. In Theorem 12 even though $r = 4$ our 4-manifold $I(\lambda)$ is compact.

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