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NONREDUCED ROOT SYSTEMS AND SOME UNIPOTENT SUBGROUPS OF ORTHOGONAL GROUPS

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1. Introduction

The set of roots of a semisimple Lie algebra over the complex number field with respect to a Cartan subalgebra forms a reduced root system and some simple groups like Chevalley groups are obtainable as groups of automorphisms of such Lie algebra. There are irreducible reduced root systems of 9 different types, but exists the unique irreducible nonreduced root system of type BC_n, which is the union of reduced root systems of type B_n and C_n, cf. [3], [4].

The reduced root systems D_n, B_n, and C_n describe generators and relations of the maximal unipotent subgroups of the even orthogonal, odd orthogonal and symplectic groups, respectively, cf. [1], [7]. In this paper, by describing generators and relations, we associate the positive nonreduced root system with nonmaximal unipotent radicals of certain parabolic subgroups of even and odd orthogonal groups over all fields of characteristic other than 2. The author did the analogous work for the symplectic groups in her thesis [5].

We follow all notations from [6] except we use the minus notation, $-i$, as in [2], instead of the underlining notation, $\underline{i}$, used in [5] and [6], since $-i$ seems more reasonable than $\underline{i}$. By the $(-i,-j)$ entry of an $m \times n$ matrix M we denote the $(m-i+1, n-j+1)$ entry of M. That is, the positive number $-i$ between 1 and m is the i-th number counted from the last number m.

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2. The Nonreduced Root System BC_n

The nonreduced root system can be obtained from a reduced root system by omitting the condition that the only multiples $\pm\alpha$ of each root α can be roots again. Let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ be the usual orthonormal vectors for n-dimensional real euclidean space. Then the positive nonreduced root system of type BC_n consists of the following 4 different types of roots denoted by

$$R_{i,j} = \varepsilon_i - \varepsilon_j, \quad R_{i,-j} = \epsilon_i + \epsilon_j, \quad R_{i,-i} = 2\epsilon_i, \quad \text{and} \quad R_{i,n+1} = \epsilon_i,$$

and $R_{i,i+1} = R_i$ are fundamental roots, where $1 \leq i < j \leq n$, $1 \leq i \leq n$. Note that $R_{i,n+1} = R_i + R_{i+1} + \cdots + R_n$ and $R_{i,-i} = 2R_{i,n+1}$.

The nonreduced root system has 3 different root lengths, unlike the reduced root systems having at most two different root lengths. So we may call roots $R_{i,j}$ and $R_{i,-j}$ of length $\sqrt{2}$ as medium roots, $R_{i,-i}$ of length 2 as long roots and $R_{i,n+1}$ of length 1 as short roots. Then the medium roots of BC_n are also the short roots of C_n and long roots of B_n, respectively. We have an analogous notion of the height $ht(R)$ of a root R in the positive nonreduced root system of type BC_n, and the roots are ordered as usual so that $ht(R) \leq ht(S)$ whenever $R < S$.

Since we use R_n as the fundamental short root $R_{n,n+1}$ in this paper, while applying the results of [6], we must always write the fundamental long root R_n in [6] as $R_{n,-n}$ in an unabbriviated notation and interpret it as a long root of height 2. In [6], where we associated some nonmaximal unipotent subgroup with the reduced root system of type C_n, the fundamental long root R_n has always been identified with $R_{n,-n}$ instead of $R_{n,n+1}$, even though the two positive integers $-n$ and $n+1$ were identical.

3. The unipotent subgroup U

Let K be a field of characteristic not equal to 2. Let V be a $2l$ $(2l+1)$ dimensional vector space over K and let J be the matrix representation of a nonsingular symmetric bilinear form on V with respect to a suitable basis, say

$$J = \begin{bmatrix} 0 & J \\ J & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & 0 & J \\ 0 & 1 & 0 \\ J & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & 0 \end{bmatrix} = \sum_i e_{i,-i}.$$

Then the even orthogonal group $G = O(2l, K)$ (the odd orthogonal group $G = O(2l+1, K)$) consists of all nonsingular matrices A such that ${}^T\!AJA = J$.

Throughout this paper, we consider the unipotent subgroup U of G as the unipotent radical of a parabolic subgroup of G which is the stabilizer of the nonmaximal flag of totally isotropic subspaces V_i of V, $1 \leq i \leq n$, satisfying the following conditions: $dimV_i/V_{i-1} \geq 2$ for all $i = 1, 2, \cdots, n$ to generate nontrivial elements corresponding to all long roots $R_{i,-i}$ and for $O(2l, K)$, $dimV_n \neq l$, *i.e.*, $dimV_n < l$ to generate nontrivial elements corresponding to all short roots $R_{i,n+1}$, too. Obviously, for $O(2l+1, K)$, it does not matter whether $dimV_n$ is equal to l or not.

Let A be any element contained in U. Then A is a $(2n+1) \times (2n+1)$ block matrix written as $I + \sum_{s<t} A_{s,t}E_{s,t}$ ($A_{s,t}E_{s,t}$ denoting the matrix $A_{s,t}$ in the (s,t) block and not matrix multiplication), which is upper triangular and whose diagonal blocks are the identity matrices. Gathering some block entries together we have a matrix A in U as

$$I + \sum_{1 \leq i < j \leq n} \{(A_{i,j}E_{i,j} + A_{-j,-i}E_{-j,-i}) + (A_{i,-j}E_{i,-j} + A_{j,-i}E_{j,-i})\}$$
$$+ \sum_{1 \leq i \leq n} \{(A_{i,n+1}E_{i,n+1} + A_{n+1,-i}E_{n+1,-i}) + A_{i,-i}E_{i,-i}\},$$

where (i,j) block entry $A_{i,j}$ is an $m_i \times n_{j-1}$ matrix over K with $m_i = n_{i-1} = dimV_i/V_{i-1}$ for each $1 \leq i, j \leq n$ and the sizes of all the other block entries are determined accordingly. Note the followings: (i) For all i, $m_i \geq 2$. (ii) $m_{n+1} = n_n = 2(l - dimV_n)$ is even, say $2y$, for even orthogonal groups and odd, $2y+1$, for odd orthogonal groups. (iii) $n+1 = -(n+1)$ as a number between 1 and $2n+1$.

Therefore, comparing with the case of [6], we now have just one more center($n+1$-th) column and row which will be related to short roots in the next section.

4. The association of BC_n with U

While ${}^T\!A$ is the ordinary transpose of a matrix A with respect to the main diagonal, by ${}^J\!A$ we denote the transpose of A with respect to the minor diagonal. In fact, ${}^J\!A$ is just the matrix multiplication $J\,{}^T\!AJ$, where J is a square matrix with 1 on the minor diagonal and o elsewhere. If $A = -{}^J\!A$, then A is a skew symmetric matrix with respcet to the minor diagonal and has all zeroes in the minor diagonal entries.

We accept all medium and long root elements of U as follows:

$$X_{i,j}(A) = I + AE_{i,j} - {}^J\!AE_{-j,-i},$$

$$X_{i,-j}(A) = I + AE_{i,-j} - {}^J\!AE_{j,-i},$$

$$X_{i,-i}(A) = I + AE_{i,-i}, \quad \text{where} \quad A = -{}^J\!A,$$

for any matrices A of suitable sizes, where $1 \le i < j \le n$, $1 \le i \le n$.

Now we see that U contains any element of the following form related to each short root $R = R_{i,n+1}$ denoted by $X_{R_{i,n+1}}(A, C) = X_{i,n+1}(A, C)$ which is

$$I + AE_{i,n+1} - {}^J\!AE_{n+1,-i} + CE_{i,-i},$$

where $-A\,{}^J\!A = C + {}^J C$ for an $m_i \times n_n$ matrix A and an $m_i \times m_i$ square matrix C over K.

In order to factor out the part corresponding to long root from $X_{i,n+1}$ (A, C), we take an upper triangular matrix A^* with respect to the minor diagonal such that the following equality holds:

$$X_{i,n+1}(A, C) = X_{i,n+1}(A, A^*)X_{i,-i}(C - A^*),$$

where A^* only depends on A. Indeed, if $-A\,{}^J\!A = [b_{s,t}] = \sum_{s,t} b_{s,t}e_{s,t}$ is a matrix whose (s,t) entry is $b_{s,t}$, then we take A^* as

$$\sum_{s<-t} b_{s,t}e_{s,t} + \sum_{s=-t} \frac{1}{2} b_{s,t}e_{s,t}.$$

Due to corollary 4.3, we call $X_{i,n+1}$ (A, A^*) the root element corresponding to short root $R_{i,n+1}$ or the short root element and also denote by $X_{i,n+1}(A)$. Also let $Mat(R,K)$ be the set of all matrices A such that each root element $X_R(A)$ makes sense. Note that each $Mat(R,K)$ is the set of all matrices of suitable sizes with no restriction except for long roots.

All the nontrivial commutator products of elements of U are obtained from the following products related to short roots in addition to the ones in [6]: for $1 \le i, j \le n$,

$$[X_{i,n+1}(A,C); X_{j,n+1}(B,D)] = \begin{cases} X_{i,-j}(-A\,{}^J\!B) & \text{if } i < j, \\ X_{i,-i}(-A\,{}^J\!B + B\,{}^J\!A) & \text{if } i = j, \\ X_{j,-i}(B\,{}^J\!A) & \text{if } i > j, \end{cases}$$

$$[X_{i,j}(A); X_{j,n+1}(B,D)] = X_{i,n+1}(AB, A\,{}^J\!D\,{}^J\!A)X_{i,-j}(A\,{}^J\!D).$$

Also note that $X_{i,n+1}(A,C)X_{i,n+1}(B,D) = X_{i,n+1}(A+B, C+D-AB)$.

In order to associate the nonreduced root system with our unipotent radical U, we expect the following theorem 4.1 and corollary 4.2, since a long root $R_{i,-i}$ is the twice of a short root $R_{i,n+1}$:

THEOREM 4.1. *Each long root subgroup $X_{i,-i}$ of U is generated by short root elements $X_{i,n+1}(A)$, where $1 \le i \le n$.*

Proof. Let $X_{i,-i}(A)$ be any element in the root subgroup $X_{i,-i}$. Then $A = [a_{s,t}] = \sum_{s,t} a_{s,t}e_{s,t}$ is an $m_i \times m_i$ square matrix such that $A = -{}^J\!A$. For each $1 \le s \le m_i$, take B_s as the $m_i \times n_n$ matrix whose last column is the transpose of the s-th row matrix of A with respect to the minor diagonal and other entries are all zeroes and consider $e_{s,1}$ as the $m_i \times n_n$ matrix whose $(s,1)$ entry is 1 and other entries are all zeroes so that

$$\begin{aligned}
&\prod_{1\le s\le m_i} \left[X_{i,n+1}(-\frac{1}{2}e_{s,1}); X_{i,n+1}(B_s)\right] \\
&\quad = \prod_{1\le s\le m_i} X_{i,-i}\left(\frac{1}{2}e_{s,1}\,{}^J\!B_s - \frac{1}{2}B_s\,{}^J\!e_{s,1}\right) \\
&\quad = X_{i,-i}\left(\frac{1}{2}\sum_{1\le s\le m_i}(e_{s,1}\,{}^J\!B_s - B_s\,{}^J\!e_{s,1})\right)
\end{aligned}$$

$$= X_{i,-i}\left(\frac{1}{2}A - \frac{1}{2}\,{}^{J}\!A\right) = X_{i,-i}(A).$$

For $O(2l,K)$, note that $X_{i,n+1}(-\frac{1}{2}e_{s,1}) = X_{i,n+1}(-\frac{1}{2}e_{s,1},0)$ and that $X_{i,n+1}(B_s) = X_{i,n+1}(B_s,0)$, since $n_n \geq 2$.

COROLLARY 4.2. *Each long root subgroup $X_{i,-i}$ of U is the commutator subgroup of the short root subgroup $X_{i,n+1}$, for $1 \leq i \leq n$.*

For long and medium roots R, the corresponding root subgroup X_R is just the set of all root elements $X_R(A)$ for all $A \in \mathrm{Mat}(R,K)$. But for short roots, we are in a different situation and obtain the following corollary:

COROLLARY 4.3. *Each short root subgroup $X_{i,n+1}$ of U is precisely the set of all elements $X_{i,n+1}(A,C)$ and also the set of all root elements $X_{i,n+1}(A,A^*)$ modulo $X_{i,-i}$, for $1 \leq i \leq n$.*

Because of commutator products, we have the unique expression of $A = I + \sum_{i<j} A_{i,j}E_{i,j}$ of U of the form, for some $B_{i,j} \in \mathrm{Mat}(R_{i,j},K)$,

$$X_{R_1}(A_{1,2})X_{R_2}(A_{2,3})\cdots X_{R_n}(A_{n,n+1})X_{R_{1,3}}(B_{1,3})X_{R_{2,4}}(B_{2,4})\cdots$$

$$X_{R_{n-1,n+1}}(B_{n-1,n+1})X_{R_{n,-n}}(B_{n,-n})X_{R_{1,4}}(B_{1,4})\cdots X_{R_{1,-1}}(B_{1,-1}),$$

where R_1, R_2, $\cdots$, $R_{1,-1}$ are all positive roots in the nonreduced root system arranged in increasing order. Furthermore we expect

THEOREM 4.4. *U is generated by the root elements $X_R(A)$ corresponding to all fundamental roots R in the nonreduced root system for $A \in Mat(R,K)$.*

proof. By the proof of proposition 2.1 in [6], we only need to generate the root elements corresponding to the roots $R_{n,-n}$ and $R_{i,n+1}$ for $1 \leq i \leq n-1$. The other roots work out as in [6]. For the long root $R_{n,-n}$, we have already proved in theorem 4.1. Suppose $X_{i,n+1}(A) = X_{i,n+1}(A,A^*)$ is any short root element for an $m_i \times n_n$ matrix $A = [a_{s,t}] = \sum a_{s,t}e_{s,t}$. Let A_s be the $m_i \times n_n$ matrix whose s-th row is the same as that of A and other entries are all zeroes for $1 \leq s \leq m_i$.

For $O(2l+1, K)$, if $dim V_n < l$, *i.e.*, $n_n = 2y+1 \geq 3$, then we consider as follows:

$$A = \sum_{1 \leq s \leq m_i} A_s, \quad \text{and} \quad A_s = A_s^1 + A_s^2 + A_s^3,$$

where

$$A_s^1 = \sum_{1 \leq t \leq y} a_{s,t} e_{s,t}, \quad A_s^2 = \sum_{-y \leq t \leq -1} a_{s,t} e_{s,t}, \quad \text{and} \quad A_s^3 = a_{s,y+1} e_{s,y+1}.$$

Then we have

$$X_{i,n+1}(A, A^*) = \prod_{1 \leq s \leq m_i} X_{i,n+1}(A_s, A_s^*).$$

Now, for $k = 1, 2, 3$, we choose $e_{s,1} \in \mathrm{Mat}(R_{i,i+1}, K)$ and $B_s^k \in \mathrm{Mat}(R_{i+1,n+1}, K)$ such that $A_s^k = e_{s,1} B_s^k \in \mathrm{Mat}$ $(R_{i,n+1}, K)$. Then we have

$$\prod_{1 \leq k \leq 3} \left[X_{i,i+1}(e_{s,1}); X_{i+1,n+1}(B_s^k, B_s^{k*})\right] X_{i,-(i+1)}(-e_{s,1}{}^J(B_s^{k*}))$$

$$= X_{i,n+1}(A_s^1, 0) X_{i,n+1}(A_s^2, 0) X_{i,n+1}(A_s^3, -\frac{1}{2}(a_{s,y+1})^2 e_{s,-s})$$

$$= X_{i,n+1}(A_s, -A_s^1\, {}^J(A_s^2) - \frac{1}{2}(a_{s,y+1})^2 e_{s,-s}) = X_{i,n+1}(A_s, A_s^*).$$

In the above equalities, note that $B_s^{k*} = 0$ for $k = 1, 2$ and that $(A_s^1 + A_s^2)\, {}^J(A_s^3) = 0$.

For $O(2l+1, K)$, if $dim V_n = l$, *i.e.*, $n_n = 2y+1 = 1$, then we may consider A_s as A_s^3, just a column matrix. For $O(2l, K)$, consider $A_s = A_s^1 + A_s^2$, since $n_n = 2y \geq 2$.

Therefore, the positive nonreduced root system of type BC_n can be associated with some nonmaximal unipotent subgroups of even and odd orthogonal groups over a field of characteristic not equal to 2.

REMARK. For $O(2l, K)$, we note that root subgroups $X_{i,-i}$ and $X_{i,n+1}$ are generated by elements of the form $X_{i,n+1}(A, 0) = I + AE_{i,n+1} - {}^JAE_{n+1,-i}$, where $A\,{}^JA = 0$.

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